Unit 13

Non-linear differential equations

Introduction

In this unit we have a final look at differential equations. We considered the solution of several types of first-order differential equation in Unit 2, including examples of non-linear equations. However, the other units devoted to differential equations considered only linear equations (in Unit 3 we looked at linear constant-coefficient second-order differential equations, in Unit 6 we studied simultaneous linear ordinary differential equations, and in Unit 12 we considered linear partial differential equations).

One reason for the emphasis on linear differential equations is that non-linear equations are difficult to solve. Usually they do not have a solution that can be expressed in terms of standard functions. But these equations arise in many interesting investigations in science, engineering and economics. It is therefore important to be able to understand their solutions. This unit describes approaches that give *approximate* or *qualitative* information about solutions, rather than trying to find the solution itself. This information may be in the form of a diagram or an approximate expression. In practice *numerical solutions* using a computer are also used, but they are outside the scope of this module.

In this unit we consider pairs of first-order non-linear differential equations involving two variables. All of the models that appear in this unit can be represented as pairs of differential equations of the form

$$\frac{dx}{dt} = u(x,y), \quad \frac{dy}{dt} = v(x,y), \tag{1}$$

where the functions u and v depend on x and y but not on t. Systems of this form, where t does not appear explicitly in the right-hand side, are said to be **autonomous** (and conversely, if the equations contained functions such as $u(x,y,t)$ and $v(x,y,t)$, they would be described as *non-autonomous*).

In this unit we use examples relating two interacting populations of animals, one a predator and the other its prey. An example is given by the *Lotka–Volterra* pair of equations

$$\dot{x} = kx\left(1 - \frac{y}{Y}\right), \quad \dot{y} = -hy\left(1 - \frac{x}{X}\right), \tag{2}$$

These equations are non-linear because of the xy terms on the right-hand sides.

where h, k, X and Y are known positive constants, and $x = x(t)$ and $y = y(t)$ represent the two population sizes at time t. A derivation of the Lotka–Volterra equations will be given in Section 1.

While describing populations of animals is a vibrant area of science, a more typical application of non-linear differential equations is to describe the motion of mechanical systems. Such systems can range in complexity from a stone launched by a slingshot to a system of planets and asteroids orbiting a star. This unit will discuss the motion of a rigid pendulum as a simple example of a mechanical system.

The equation of motion for the undamped motion of a rigid pendulum is the second-order differential equation

This equation is non-linear because of the $\sin x$ term.

$$\ddot{x} + \omega^2 \sin x = 0, \tag{3}$$

where ω is a constant and $x = x(t)$ is the angle in radians that the pendulum makes with the downward vertical at time t. As might be expected, this equation is in the form of Newton's second law, $m\ddot{x} = F(x)$ (where the force is $F(x) = -m\omega^2 \sin x$). But equation (3) does not appear to be in the same form as equations (1). To make the connection clear, consider the following system of equations:

$$\dot{x} = y, \quad \dot{y} = -\omega^2 \sin x. \tag{4}$$

Note that this pair of differential equations is in the form of equations (1), with $u(x, y) = y$ and $v(x, y) = -\omega^2 \sin x$. Also, observe that if you differentiate the first equation of (4) and substitute in the second, you recover equation (3). There is a general principle here: you can convert a single second-order differential equation into two coupled first-order equations of the form of equations (1). By means of this approach, the methods developed in this unit can also be applied to second-order differential equations.

Non-linear equations and chaos

Sometimes non-linear differential equations have solutions that are relatively easy to understand, even if you cannot find an exact mathematical expression. For example, a set of equations such as (1) may have solutions in which $x(t)$ and $y(t)$ approach an *equilibrium point* (also known as a *fixed point*), so that $x(t) \to X$ and $y(t) \to Y$ as $t \to \infty$, for two constants X and Y.

Non-linear equations may also produce complicated and apparently erratic behaviour that defies conventional mathematical descriptions. An example is shown in Figure 1. This is a plot of a solution of a system of three coupled first-order differential equations, called the *Lorenz equations*:

$$\dot{x} = \sigma(y - x), \quad \dot{y} = x(\rho - z) - y, \quad \dot{z} = xy - \beta z, \tag{5}$$

where the constants are $\sigma = 28$, $\rho = 10$ and $\beta = 8/3$ in this case.

This type of motion is called *chaotic*. The solution $(x(t), y(t), z(t))$ spends most of the time spiralling around two surfaces, but it occasionally jumps between these surfaces. Despite the fact that the equation of motion is known precisely, these jumps occur at times that are very hard to predict. The possibility that simple systems of differential equations could produce highly unpredictable motion was not widely appreciated until late in the twentieth century.

This discovery has had a profound influence on research in the sciences, and has stimulated the development of a new mathematical discipline called *dynamical systems theory*.

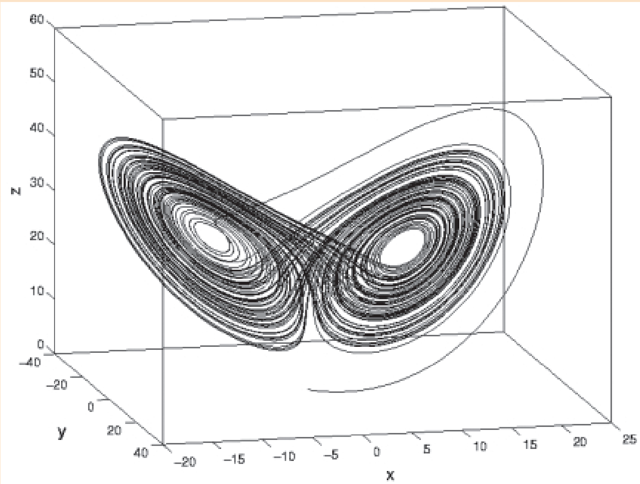


Figure 1 A solution of the Lorenz equations (5), obtained using a computer; as t increases, the solution $(x(t), y(t), z(t))$ follows a trajectory that cannot be described by standard mathematical functions

To see examples of chaos, you need to have more than two variables (as in equations (5)), or you need to consider non-autonomous equations. Although chaos does not occur in solutions of equations (1) for any choice of the functions u and v, the methods developed in this unit can also be applied to analyse chaotic motion.

For equations of form (1), we consider the paths defined by$(x(t), y(t))$ in the xy-plane. A very powerful approach starts out by seeking a constant solution (which describes an equilibrium state of the system). For example, the Lotka–Volterra equations (2) have a constant solution $x(t) = X$, $y(t) = Y$, which describes a situation where the two populations are in equilibrium. Near such a solution, you will see that some useful information can be obtained by replacing the original non-linear equations by linear approximations to the differential equations. The equilibrium states and the behaviour of the system when it is nearly in equilibrium play a major part in obtaining a qualitative overview of the behaviour of the model. This approach of determining *equilibrium points* or *fixed points* of a non-linear system, and then using *linearised equations* to study motion in the neighbourhood of equilibrium points, is the most powerful available tool for analysing systems of non-linear differential equations.

We use the notations x or $x(t)$, $\dot{x}$ or $\dot{x}(t)$, etc., interchangeably to suit the context.

Study guide

We begin in Section 1 with the Lotka–Volterra equations, which apply to a pair of interacting populations. In Section 2 we see how these equations can be *linearised* (that is, approximated by linear equations) near an equilibrium state. The resulting system of linear differential equations was discussed in Unit 6, but here, in Section 3, we use graphical representations of the solutions near an equilibrium state to get qualitative information about solutions of the original non-linear equations. Section 4 looks at models for the motion of a rigid pendulum: second-order differential equations are transformed to systems of first-order equations, and the techniques developed earlier in the unit are applied to find and interpret graphical solutions.

You need to be familiar with the techniques developed in Unit 5 for finding eigenvalues and eigenvectors, and in Unit 6 for the solution of systems of first-order linear differential equations.

1 Modelling populations of predators and prey

In this section we develop models for populations of a predator and its prey. A predator population depends for its survival on there being sufficient prey to provide it with food. Intuition suggests that when the number of predators is low, the prey population may increase quickly, and that this will eventually result in an increase in the predator population. On the other hand, a large number of predators may diminish the prey population until it is very small, and this will eventually lead to a collapse in the predator population. Our mathematical models will need to reflect this behaviour.

It is not possible to find exact solutions to all such models, so we will introduce you to a geometric approach, based on the notion that a point $(x, y) = (x(t), y(t))$ in a plane may be used to represent two populations $x = x(t)$ and $y = y(t)$ at time t. As t increases, the point $(x(t), y(t))$ traces out a path that represents the variation of both populations with time.

1.1 Exponential growth of a single population

Before we consider a system of two interacting populations, we first develop a simple continuous model of the growth of a single population, which is called the *exponential model*. This allows us to develop some basic concepts of population models in a simple context.

This model was originally discussed in Unit 2.

We model the population size x, which we usually simply refer to as the population x (omitting the word 'size'), as a function of time t. This

function cannot take negative values (since there are no negative populations), but we allow it to take the value zero. We normally assume that t is measured in years. We deal with a continuous model, rather than a discrete one, so the derivative $\dot{x} = dx/dt$ represents the rate of increase of the population, which we often refer to as the **growth rate** (even though when $\dot{x} < 0$ it actually represents a decay rate).

A population x can take only integer values, so we say that x is a discrete variable. Here, however, we approximate the population by a variable that can take any real value, referred to as a continuous variable.

In the exponential model, we make the assumption that the growth rate $\dot{x}$ is proportional to the current population x. (This means that if the growth rate is 20 per year when the population is 100, then the growth rate will be 40 per year when the population is 200, the growth rate will be 60 per year when the population is 300, and so on.) This assumption leads directly to the differential equation

$$\dot{x} = kx \quad (x \geq 0), \tag{6}$$

where k is a constant. If k is positive, then x is an increasing function of time, while if k is negative, then x is a decreasing function of time.

Exercise 1

Under what circumstances is it reasonable to assume that the growth rate of a population is proportional to the current population? (This is an open-ended question that is primarily about modelling rather than mathematics.)

Equation (6) can be solved explicitly. Choosing a value for the population at time $t = 0$, for example $x(0) = x_0$, gives the solution

$$x(t) = x_0 e^{kt} \quad (x_0 \geq 0). \tag{7}$$

The nature of this solution depends on the sign of the constant k. If $k = 0$, the population remains constant in time. If $k > 0$, the population increases exponentially, and if $k < 0$, the population decreases exponentially. The two situations shown in Figure 2 are described, respectively, as *exponential growth* and *exponential decay* (although we could also say that the latter exhibits 'negative growth'). For this reason we refer to equation (6) as the **exponential differential equation**, or, when applied to a population, the **exponential model**.

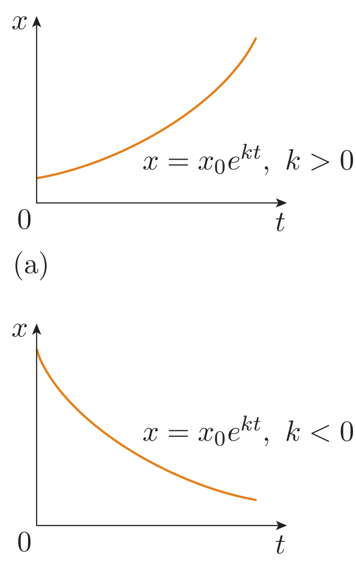


Figure 2 Behaviour of a population modelled by equation (7). For $k > 0$ there is exponential growth, and for $k < 0$ there is exponential decay.

The **proportionate growth rate** $\dot{x}/x$ represents the rate of increase of the population per unit of the current population, and may be considered as the difference between the birth and death rates *per head of population*. It may be positive (for a growing population in which the birth rate exceeds the death rate), negative (for a declining population in which the death rate exceeds the birth rate) or zero (for a static population in which the birth and death rates are equal). In the case of the exponential model (6), we have $\dot{x}/x = k$, so the proportionate growth rate is constant. (We previously assumed that $x \geq 0$. While talking about proportionate growth rate, we exclude the possibility that x takes the value zero.)

Exercise 2

Can you suggest any reason why, in the case $k > 0$, the assumption that the proportionate growth rate $\dot{x}/x$ is a constant is unrealistic for real populations?

The exponential model is fairly accurate for many populations in a state of rapid increase, but it is reasonable only over a restricted domain of time. As a model of the behaviour of increasing populations over longer periods of time, the exponential model is clearly unsatisfactory because it predicts unbounded growth. In Unit 2 we mentioned one way of addressing this deficiency when we considered the *logistic equation*: this allows for the fact that the rate of population growth decreases as the population increases, due to competition for resources. However, in this unit we now adopt a more sophisticated approach by explicitly including variables representing the populations of two species, a population and its prey.

1.2 Motion in phase space

In the rest of this section we will be concerned with developing models for populations of rabbits (the prey) and foxes (the predators). Our purpose is to determine how these populations evolve with time. At a particular time t, we suppose that these populations are $x(t)$ and $y(t)$, respectively. Before discussing the differential equations that determine these functions, we introduce some geometrical language that will help us to discuss equations of motion written in the form (1).

x is the prey population; y is the predator population.

At any given time, we represent this system by a point in the xy-plane. We call this two-dimensional space the **phase space** (or **phase plane**) for the system, and the point is called the **phase point**. We can think of x and y as the coordinates of the position vector $\mathbf{x}$ of the phase point. The evolution of the two populations can then be represented by a path, called a **phase path** (or **phase trajectory**), in the phase space, as shown in Figure 3. Here, the direction of the arrows indicates the direction in which the phase point $(x(t), y(t))$ moves along the phase path with increasing time. However, this type of representation does not show how quickly or slowly the point moves along the path.

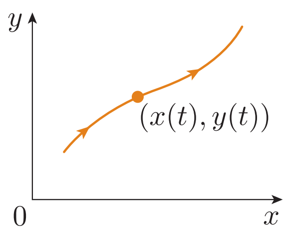


Figure 3 A phase path in phase space

The position of the phase point at time t is represented by the column vector

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},$$

and the corresponding velocity of the phase point is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix}.$$

Using equations (1), this can be expressed as

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}.$$

The functions $u(x,y)$ and $v(x,y)$ can be regarded as representing a *vector field*

In this context, the term vector field simply means a field with two components.

$$\mathbf{u}(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix},$$

and equations (1) can then be written in the vector form

$$\dot{\mathbf{x}} = \mathbf{u}(x,y). \tag{8}$$

This is an equation of motion for the phase point: the velocity of the phase point, $\dot{\mathbf{x}}$, is equal to $\mathbf{u}(\mathbf{x})$, that is, to the value of a vector field $\mathbf{u}$ evaluated at the position of the phase point $\mathbf{x} = (x,y)$. As the phase point moves along the phase path, the velocity $\dot{\mathbf{x}}$ of the phase point is always tangential to the phase path. We therefore reach the following conclusion.

The vector field $\mathbf{u}(x,y)$ is tangential to the phase path at (x,y).

This geometrical concept is a powerful method for getting insight into the behaviour of systems of differential equations that cannot be solved exactly.

Multiple meanings of phase

The word 'phase' crops up in many different areas of science with no obvious connections. In this unit we have defined *phase point*, *phase path* and *phase space*. In Unit 3 we wrote an equation for an oscillation in the form $x(t) = A\sin(\omega t + \phi)$, and referred to ϕ as the *phase constant* of the oscillation. In physics and chemistry you will also find discussions of *phase transitions*, which are abrupt changes of the properties of a substance in response to changes of temperature. Don't seek a deep connection: there isn't one.

The word *phase* is derived from a Greek word with a meaning similar to 'appearance', so it is natural that it was adopted independently by different branches of science. Its use in discussing oscillations arose from the fact that the phases of the moon are a periodic phenomenon.

1.3 A first predator–prey model: sketching phase paths

In our mathematical models of rabbit and fox populations we make the following assumptions.

- There is plenty of vegetation for the rabbits to eat.
- The rabbits are the only source of food for the foxes.
- An encounter between a fox and a rabbit contributes to the fox's larder, which leads directly to a decrease in the rabbit population and indirectly to an increase in the number of foxes.

More sophisticated models also take account of the fact that there is a time delay for the population to increase due to animals breeding, or for the population to decrease due to starvation.

We may, for convenience, measure the populations in hundreds or thousands, as appropriate, so that we are able to use quite small numbers in our models. The population of rabbits will be described by a function $x(t)$, and the population of foxes by $y(t)$.

We begin with a very simple model for the two populations; this generalises the exponential model of the previous subsection, which applies to a single population. Our simple model has the advantage that we can easily find an explicit solution, and this will help us to explore the geometric approach.

As a first model, we assume that the populations are evolving independently (perhaps on separate islands). Because there are *no interactions*, the populations may be modelled by the pair of equations

$$\dot{x} = kx, \quad \dot{y} = -hy \quad (x \geq 0,\ y \geq 0), \tag{9}$$

where k and h are positive constants. The conditions $x \geq 0$, $y \geq 0$ in parentheses apply here, as well as in similar situations throughout the unit, to the first equation $\dot{x} = kx$, as well as the second equation $\dot{y} = -hy$.

The first equation models a colony of rabbits not affected by the predation of foxes, growing exponentially according to a rule of the form

$$x(t) = x_0 e^{kt}, \tag{10}$$

When $t = 0$, $x = x_0$.

where x_0 is a positive constant representing the initial rabbit population at $t = 0$.

Exercise 3

Determine a formula for the population $y(t)$ of foxes. How would you interpret this solution for the population of foxes?

Equations (9) form a system of linear differential equations similar to those that you met in Unit 6. Let us now consider the solution of these equations from the geometrical viewpoint of motion in phase space, which we discussed above. Using vector notation, the pair of populations may be represented by the vector $\mathbf{x} = [x \quad y]^T$. The system of equations (9) then becomes the vector equation

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \mathbf{u}(x, y) = \begin{bmatrix} kx \\ -hy \end{bmatrix}. \tag{11}$$

Recall also that $\mathbf{u}(x, y)$ is tangential to the phase path describing a particular solution of $\dot{\mathbf{x}} = \mathbf{u}(x, y)$ at the point (x, y).

This suggests a geometric way of finding a particular solution of equation (11). On a diagram, we draw a selection of arrows representing the directions of the vector field $\mathbf{u}(x, y)$ at a selection of points (x, y). Since the magnitudes of $\mathbf{u}(x, y)$ may vary considerably and so make the diagram difficult to interpret, we often use arrows of a fixed length. Then, choosing a particular starting point (x_0, y_0), we follow the directions of the arrows to obtain a phase path corresponding to a particular solution.

An exception, which we discuss later, occurs when $\dot{x} = \dot{y} = 0$ at (x_0, y_0), so $\mathbf{u}(x_0, y_0) = \mathbf{0}$ and there is no tangent vector to follow.

To see how this works, consider equation (11) with $k = 1$ and $h = 1$. In this case the vector field $\mathbf{u}(x, y)$ is given by

$$\mathbf{u}(x, y) = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

Figure 4 shows the directions of this vector field at a selection of points in the xy-phase space. For example, the red arrow centred at $x = 1$, $y = 3$ points in the direction of $[1 \quad -3]^T$, while the blue arrow centred at $x = 3$, $y = 1$ points in the direction of $[3 \quad -1]^T$. The phase path at any point (x, y) is tangential to $\mathbf{u}(x, y)$, so the continuous curve in Figure 4 is a reasonable estimate of a phase path representing a particular solution of equation (11) (with $k = h = 1$). This phase path is in broad agreement with the results of equation (10) and Exercise 3: it shows that a decrease in y (the fox population) is accompanied by an increase in x (the rabbit population).

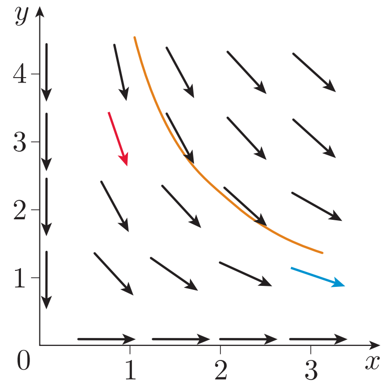


Figure 4 A map of the vector field $\mathbf{u}(x, y)$ for equation (11) with $k = h = 1$ ($x \geq 0$, $y \geq 0$)

The methods that we have just used to analyse a simple population model can be applied more generally. To explore these methods, we now widen the discussion and look at similar systems that do not arise from populations. In this broader context, we allow x and y to take positive or negative values.

Example 1

(a) Using arrows of a fixed length, sketch a map of the vector field

$$\mathbf{u}(x, y) = \begin{bmatrix} 0.2x \\ 0.3y \end{bmatrix}$$

for $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$. Sketch a few phase paths for various initial conditions.

(b) Write down the system of differential equations $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x})$ corresponding to this vector field. Find the general solutions of this system of equations, and hence obtain an equation for y in terms of x for each path in the xy-plane represented by this general solution.

(c) Use your answer to part (b) to sketch a sample of the paths in the xy-plane that represent typical particular solutions.

(d) Comment on the relationship between the vector field map in part (a) and the phase paths sketched in part (c).

Solution

(a) We use values of $\mathbf{u}(x, y)$ to construct Figure 5. For example, the red arrow at $x = 1$, $y = 1$ points in the direction of $[0.2 \quad 0.3]^T$, while the blue arrow at $x = -2$, $y = 3$ points in the direction of $[-0.4 \quad 0.9]^T$. In each case, the arrow at (x, y) represents a vector of a fixed length parallel to $\mathbf{u}(x, y)$, and therefore indicates the direction of a phase path through the point (x, y). A few phase paths have been (tentatively) sketched on this diagram.

It is time-consuming to draw vector field maps by hand, but computers can help. The subsequent sketching of phase paths is not a precisely defined process, and different people may get slightly different results.

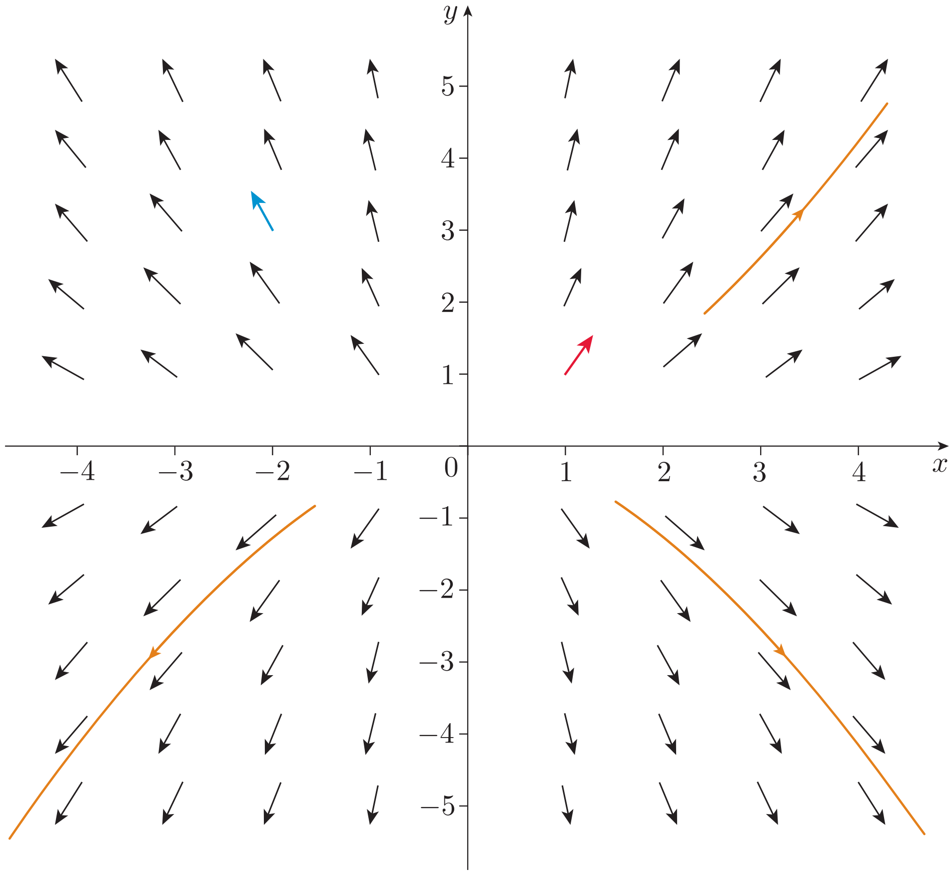


Figure 5 A map of the vector field and some phase paths

(b) We have

$$\mathbf{u}(x,y) = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0.2x \\ 0.3y \end{bmatrix},$$

so the system of equations is

$$\dot{x} = 0.2x, \quad \dot{y} = 0.3y.$$

Each of these equations can be solved separately. The general solutions are

$$x(t) = x_0 e^{0.2t}, \quad y(t) = y_0 e^{0.3t}.$$

To obtain the equation for the paths in the form of a function $y(x)$, we must eliminate t from $x = x_0 e^{0.2t}$ and $y = y_0 e^{0.3t}$.

Cubing the first equation and squaring the second gives $x^3 = x_0^3 e^{0.6t}$ and $y^2 = y_0^2 e^{0.6t}$, so $x^3/x_0^3 = y^2/y_0^2$. Hence the equations of the paths are of the form

$$y = K|x|^{3/2},$$

for some constant K.

The modulus sign around x ensures that we do not try to obtain the square root of a negative number. The arbitrariness of K ensures that the equation represents all possible cases.

(c) As we have been able to find the equations of the paths in part (b), namely $y = K|x|^{3/2}$, we can use these to sketch the phase paths in the xy-plane. These are shown in Figure 6.

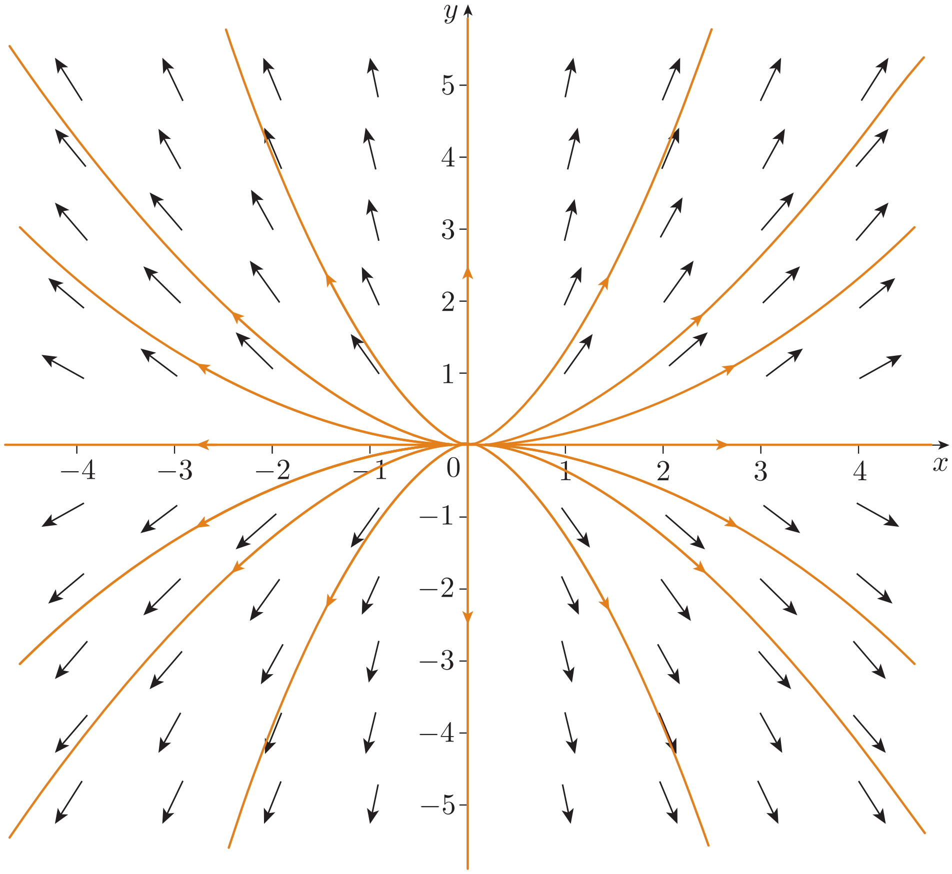


Figure 6 Phase paths superimposed over the vector field

The arrows on these paths indicate the direction of motion as time increases, and may be determined as follows.

The first of the differential equations is $\dot{x} = 0.2x$. So for positive x we have $\dot{x} > 0$, and x is an increasing function of time. Therefore in the right half-plane, the arrows on the paths point to the right. Similarly, for negative x we have $\dot{x} < 0$, and x is a decreasing function of time. Hence the arrows on the paths point to the left in the left half-plane. (Consideration of the second differential equation, $\dot{y} = 0.3y$, confirms the directions of the arrows on the paths.) The arrows on the paths along the positive and negative y-axes can be deduced from consideration of the differential equation $\dot{y} = 0.3y$. Note that the origin is also a path, corresponding to $K = 0$, but it has no time arrow associated with it.

(d) The vector field $\mathbf{u}(x, y)$ is tangential at (x, y) to the phase path passing through (x, y). The arrows in Figure 6 show the directions of the vector field at an array of points. We should therefore expect the direction of an arrow at (x, y) to be close to the direction of any phase path that passes close to (x, y). This expectation is borne out by the arrows and phase paths in Figure 6. The tentative paths sketched in Figure 5 are close to the exact paths in Figure 6.

Given a vector map of arrows at a closely-spaced set of points, it is possible to sketch the general form and directions of phase paths passing near these points. A diagram showing a selection of these phase paths in the phase plane is called a **phase diagram**. The significance of such a diagram is that each of the phase paths corresponds to a particular solution of the system of equations.

Before we leave Example 1, you may have noticed in Figure 6 that the paths radiate *outwards* from the origin in all directions. For this reason, we refer to the origin as a *source*.

A source can occur at a point other than the origin.

We now look at the phase paths for a similar system, for which

$$\mathbf{u}(x,y) = \begin{bmatrix} -0.2x \\ -0.3y \end{bmatrix}. \tag{12}$$

This system behaves in a similar fashion to the system in Example 1. The general solution is

$$x = x_0 e^{-0.2t}, \quad y = y_0 e^{-0.3t}.$$

Eliminating t gives

$$y = K|x|^{3/2},$$

as before. However, in this case, as t increases, the direction of motion along the phase paths is the opposite to that in Example 1. At any point (x, y), the vector field $[-0.2x \quad -0.3y]^T$ has the same magnitude but the opposite direction to the vector field $[0.2x \quad 0.3y]^T$. The phase diagram is therefore identical to Figure 6 except that the directions now point *towards* the origin. In a case like this, the origin is said to be a *sink*.

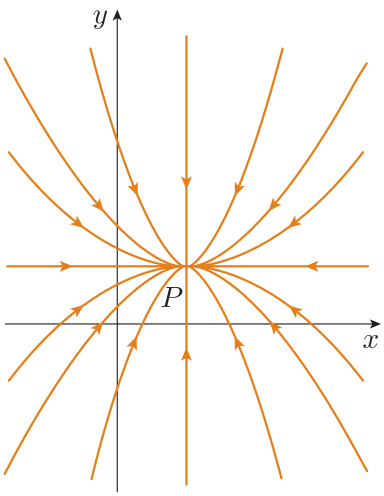


Figure 7 A sink at a point other than the origin

Sources and sinks need not always be at the origin. In general, if all the phase paths in the vicinity of any point P radiate outwards from P, then P is a **source**, and if all the phase paths in the vicinity of P converge inwards towards P, then P is a **sink** (see Figure 7).

Exercise 4

Write down the system of differential equations $\dot{\mathbf{x}} = \mathbf{u}(x, y)$ given by the vector field

This is the equation for our first model for the predator–prey populations (equation (11)) with $k = 1$ and $h = 1$, but without the restrictions $x \geq 0$, $y \geq 0$.

$$\mathbf{u}(x,y) = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

Find the general solution of this system of equations. Hence find an equation for y in terms of x for the phase paths in the xy-plane represented by this solution. Sketch some of these paths. Do any of the paths include the origin?

The phase diagram for the vector field examined in Exercise 4 is shown in Figure 8. You can see that the vast majority of paths do not radiate into or out of the origin. On these paths, a point initially travels towards the

origin, but eventually travels away from it again. The only paths that actually radiate inwards towards, or outwards from, the origin are those along the x- and y-axes. In this case we call the origin a **saddle** (because the phase paths resemble contour lines near a saddle point).

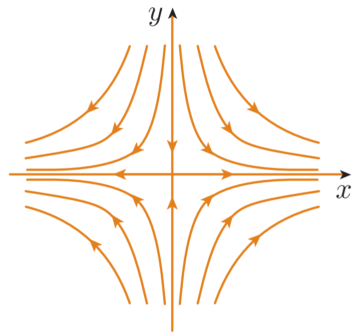


Figure 8 Phase paths near a saddle, discussed in Exercise 4

Note that the model described by equations (9) has phase paths in the form of a saddle, because the coefficients have different signs. The behaviour of the populations of rabbits and foxes illustrated in the quadrant $x \geq 0$, $y \geq 0$ of Figure 8 is what we would expect from our first model. The population x of rabbits increases without limit, as they are isolated from their predators. On the other hand, the population y of foxes decreases to zero, as they have no access to their sole source of food.

1.4 A second predator–prey model

In the previous subsection we looked at a simple model for rabbit and fox populations when there was no interaction between the two populations. This model may be reasonable when both species inhabit the same environment but with population sizes that are so small that the rabbits and foxes rarely meet. As we saw, for initial positive populations, this first model predicts that rabbits will increase without limit and foxes will die out. However, this is not what we expect for interacting populations, when encounters are inevitable.

In this subsection we look at a revised model, based on the assumption that the number of encounters between foxes and rabbits is proportional both to the population x of rabbits and to the population y of foxes. In addition to the assumptions listed at the beginning of Subsection 1.3, our revised model assumes that

- the number of encounters between foxes and rabbits is proportional to the product xy.

For a population x of rabbits in a fox-free environment, our first model for population change was given by the equation $\dot{x} = kx$, where k is a positive constant. This represents exponential growth. However, if there is a population y of predator foxes, we should expect the growth rate $\dot{x}$ of the rabbit population to be reduced. A simple assumption is that

- the growth rate $\dot{x}$ of the rabbit population decreases by a term that is proportional to the number of encounters between rabbits and foxes, i.e. by a term proportional to xy.

We revise our first model to include this extra term, so the differential equation that models the population x of rabbits is now

$$\dot{x} = kx - Axy,$$

for some positive constant A. As you will see later, it is convenient to write $A = k/Y$, for some positive constant Y, giving

$$\dot{x} = kx\left(1 - \frac{y}{Y}\right). \tag{13}$$

In a period of time δt, the change in the rabbit population is the number $kx\,\delta t$ of additional rabbits born (taking into account those dying from natural causes) minus the number $Axy\,\delta t$ of rabbits eaten.

Similarly, for a population y of foxes in a rabbit-free environment, our first model for the population change is given by the equation $\dot{y} = -hy$, where h is a positive constant. This represents exponential decay. However, if there is a population x of rabbits for the foxes to eat, we should expect the growth rate $\dot{y}$ of the fox population to increase. A simple assumption is that

- the growth rate $\dot{y}$ of the fox population increases by a term that is proportional to the number of encounters between foxes and rabbits, i.e. by a term proportional to xy.

Our revised model for the foxes is given by

$$\dot{y} = -hy + Bxy,$$

for some positive constant B. Again, it is convenient to write $B = h/X$, for some positive constant X, so that this equation becomes

$$\dot{y} = -hy\left(1 - \frac{x}{X}\right). \tag{14}$$

Together, the differential equations (13) and (14) model the pair of interacting populations.

Exercise 5

Use equations (13) and (14) to sketch the graph of the proportionate growth rate $\dot{x}/x$ of rabbits as a function of the population y of foxes, and the graph of the proportionate growth rate $\dot{y}/y$ of foxes as a function of the population x of rabbits. Interpret these graphs.

Equations (13) and (14) provided one of the first applications of mathematical models to biological ecosystems. They were proposed independently by the American biophysicist Alfred Lotka (in 1925) and by the Italian mathematician Vito Volterra (in 1926), and they are called the Lotka–Volterra equations.

The Lotka–Volterra equations

The evolution of two interacting populations x and y can be modelled by the **Lotka–Volterra equations**

As for equations (9), the conditions $x \geq 0$, $y \geq 0$ apply to both differential equations.

$$\dot{x} = kx\left(1 - \frac{y}{Y}\right), \quad \dot{y} = -hy\left(1 - \frac{x}{X}\right) \quad (x \geq 0,\ y \geq 0), \tag{15}$$

where x is the population of the prey, y is the population of the predators, and k, h, X and Y are positive constants.

The Lotka–Volterra equations are non-linear because their right-hand sides contain a term proportional to xy.

The Lotka–Volterra equations can also be written as

$$\dot{\mathbf{x}} = \mathbf{u}(x, y),$$

where $\dot{\mathbf{x}} = [\dot{x} \quad \dot{y}]^T$ and the vector field $\mathbf{u}(x, y)$ is given by

$$\mathbf{u}(x, y) = \begin{bmatrix} kx\left(1 - \dfrac{y}{Y}\right) \\ -hy\left(1 - \dfrac{x}{X}\right) \end{bmatrix}. \tag{16}$$

Exercise 6

Suppose that in equations (15), $k = 0.05$, $h = 0.1$, $X = 1000$ and $Y = 100$. Find the values of the corresponding vector field $\mathbf{u}(x, y)$ at the following points.

(a) $(0, 0)$ (b) $(0, 100)$ (c) $(500, 100)$ (d) $(1000, 0)$

(e) $(1000, 100)$ (f) $(1500, 100)$ (g) $(1000, 50)$ (h) $(1000, 150)$

Previously, we were able to find exact solutions for the pairs of differential equations that arose from our mathematical model, but for the Lotka–Volterra equations (15), no explicit solutions for $x(t)$ and $y(t)$ are available. We will therefore rely on geometrical arguments, based on vector field maps in the phase plane. For the parameters and (x, y) points given in Exercise 6, we can begin to construct a vector field map, as shown in Figure 9, where we have used arrows of fixed length.

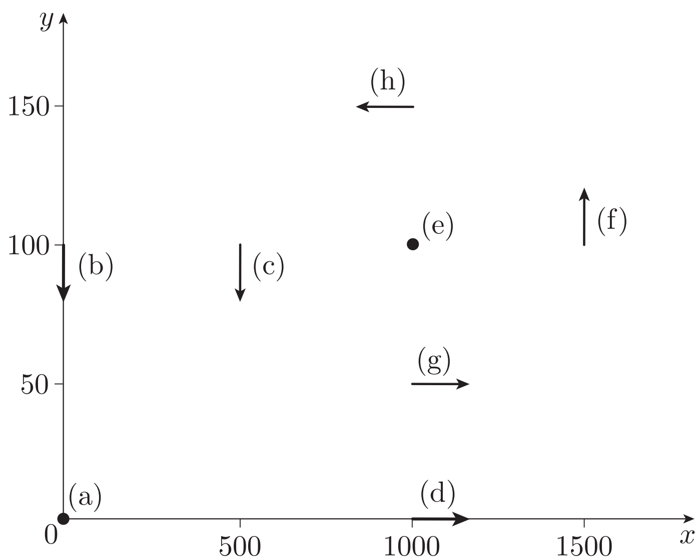


Figure 9 Directions of the vector field $\mathbf{u}(x, y)$ for the Lotka–Volterra equations as calculated in Exercise 6

Figure 10 continues this process by adding more arrows.

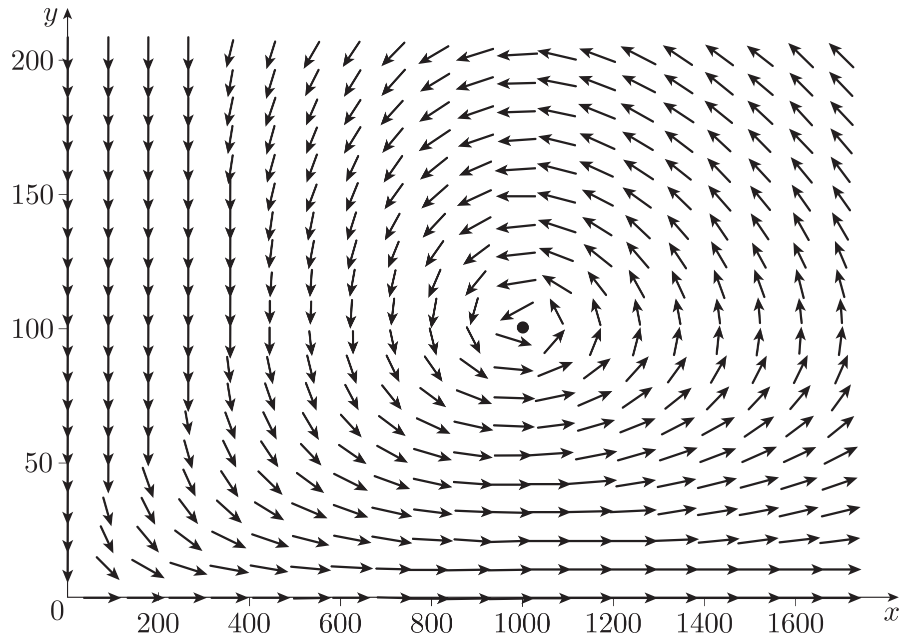


Figure 10 Directions of the vector field $\mathbf{u}(x, y)$ for the Lotka–Volterra equations at many points

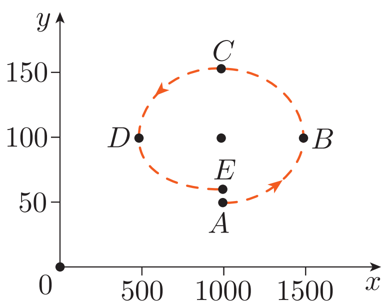


Figure 11 A phase path consistent with Figure 10 might spiral in to the fixed point

From this arrow map we can try to sketch some phase paths. Figure 11 shows one such attempt, which corresponds to a guess at a particular solution of the equations. In Figure 11 we have labelled a point A on a path, which can be taken as the initial value for a particular solution. Some other points have been marked to aid the following discussion. To interpret this guess at a solution, we think about what happens to the values of the populations as we follow the dashed path.

Initially, at the point A, there are 1000 rabbits and 50 foxes. As we follow the path, the rabbit population increases and so does the fox population, until at the point B we have reached a maximum rabbit population. As the fox population continues to rise, the rabbit population goes into decline. At C, the fox population has reached its maximum, while the rabbits decline further. After this point, there are not enough rabbits available to sustain the number of foxes, and the fox population also goes into decline. At D, the declining fox population gives some relief to the rabbit population, which begins to pick up. Finally, at E, the decline of the fox population is halted as the rabbit population continues to increase.

While it is clear from the form of the vector field that we must return close to the point A, it is impossible to decide exactly what will happen from this geometrical picture. The path could 'spiral in', as illustrated in Figure 11, it could 'spiral out', or the phase path might close on itself exactly, in which case the cycle will repeat indefinitely. In fact, we will show later (in Subsection 3.5) that for the Lotka–Volterra equations, the paths close on themselves (so that the point E in Figure 11 is the same as point A). This implies that the populations for the Lotka–Volterra model are periodic functions of time.

Exercise 7

Consider the Lotka–Volterra system of differential equations defined in equations (15), with $k = 0.05$, $h = 0.1$, $X = 1000$ and $Y = 100$, and with $x \geq 0$, $y \geq 0$.

Recall that the Lotka–Volterra equations are defined only in the quadrant $x \geq 0$, $y \geq 0$.

(a) For what values of x and y do the following hold?

(i) $\dot{x} = 0$ (ii) $\dot{x} > 0$ (iii) $\dot{x} < 0$

(b) For what values of x and y do the following hold?

(i) $\dot{y} = 0$ (ii) $\dot{y} > 0$ (iii) $\dot{y} < 0$

(c) Using your answers to parts (a) and (b), and Figures 9 and 10, sketch some more phase paths representing solutions of the system of differential equations. You may assume that for the Lotka–Volterra model, paths that return close to the starting point are closed.

2 Equilibrium and stability

2.1 Equilibrium points

Systems of differential equations such as (1) usually have solutions where both $x(t)$ and $y(t)$ are constant, taking values denoted x_e and y_e, respectively. These solutions are known as *equilibrium* or *fixed point* solutions. A particular solution $(x(t), y(t))$ may or may not approach (x_e, y_e) as $t \to \infty$; this depends on the **stability** of the equilibrium point. But equilibrium point solutions are important because they are the most easily calculated property that gives quantitative information about a pair of non-linear differential equations.

Examples of equilibrium points are the point $(0, 0)$ in Figure 8 and the point $(1000, 100)$ in Exercises 6 and 7. The equilibrium point $(1000, 100)$ corresponds to the fact that a rabbit population of 1000 and a fox population of 100 can coexist in equilibrium, not changing with time.

As you will see later, the point $(1000, 100)$ is an isolated point through which no phase path passes.

More generally, if $x(t) = C$, $y(t) = D$ is a constant solution of a system of differential equations, it follows that $\dot{x}(t) = 0$, $\dot{y}(t) = 0$, and we can use this property to find all the equilibrium points of the system.

Definition

An **equilibrium point** (or **fixed point**) of a system of differential equations

$$\dot{\mathbf{x}} = \mathbf{u}(x, y)$$

is a point (x_e, y_e) such that $x(t) = x_e$, $y(t) = y_e$ is a constant solution of the system of differential equations, i.e. (x_e, y_e) is a point at which $\dot{x}(t) = 0$ and $\dot{y}(t) = 0$.

Procedure 1 Finding equilibrium points

To find the equilibrium points of the system of differential equations

$$\dot{\mathbf{x}} = \mathbf{u}(x, y),$$

for some vector field $\mathbf{u}$, solve the equation

$$\mathbf{u}(x, y) = \mathbf{0},$$

for x and y. If the variables x and y represent populations, they must satisfy the conditions $x \geq 0$ and $y \geq 0$.

Solving $\mathbf{u}(x, y) = \mathbf{0}$ requires the solution of two simultaneous equations (generally non-linear) for the unknowns x and y.

Example 2

Find the equilibrium points for the Lotka–Volterra equations (15) for rabbit and fox populations.

Solution

Using Procedure 1, we need to solve the equation $\mathbf{u}(x, y) = \mathbf{0}$, which becomes

$$\begin{bmatrix} kx\left(1 - \dfrac{y}{Y}\right) \\ -hy\left(1 - \dfrac{x}{X}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives the simultaneous equations

$$kx\left(1 - \frac{y}{Y}\right) = 0,$$
$$-hy\left(1 - \frac{x}{X}\right) = 0.$$

From the first equation, we deduce that either $x = 0$ or $y = Y$.

As stated earlier, h, k, X and Y are *positive* constants.

If $x = 0$, the second equation reduces to $-hy = 0$, so $y = 0$ and hence $(0, 0)$ is an equilibrium point.

If $y = Y$, the second equation becomes $-hY(1 - x/X) = 0$, so $x = X$ and hence (X, Y) is an equilibrium point.

Thus there are two possible equilibrium points for the pair of populations. The first has both the rabbit and fox populations as zero, i.e. the equilibrium point is at $(0, 0)$; there are no births or deaths – nothing happens. However, the other equilibrium point occurs when there are X rabbits and Y foxes, i.e. the equilibrium point is at (X, Y), when the births and deaths exactly cancel out and both populations remain constant.

This explains our choice of constants X and Y in Subsection 1.4.

Exercise 8

Suppose that the population x of a prey animal and the population y of a predator animal evolve according to the system of differential equations

$$\dot{x} = 0.1x - 0.005xy, \quad \dot{y} = -0.2y + 0.0004xy \quad (x \geq 0,\ y \geq 0).$$

Find the equilibrium points of the system. Put these equations in the standard form of the Lotka–Volterra equations.

Exercise 9

Suppose that two interacting populations x and y evolve according to the system of differential equations

$$\dot{x} = x(20 - y), \quad \dot{y} = y(10 - y)(10 - x) \quad (x \geq 0,\ y \geq 0).$$

These are *not* Lotka–Volterra equations.

Find the equilibrium points of the system.

2.2 Dynamics close to equilibrium

In a real ecosystem it is unlikely that predator and prey populations are in perfect harmony. What if equilibrium is disturbed by a small deviation caused perhaps by a severe winter or hunting? If the number of rabbits is reduced, there would be a decreased food supply for the foxes, and the population of foxes could decrease to zero as a consequence. On the other hand, if the number of foxes is reduced, the birth rate for rabbits would then exceed their death rate, and the number of rabbits could increase (perhaps without limit in a simplified model).

If a small change or *perturbation* in the populations of rabbits and foxes from their equilibrium values, for no matter what reason, results in subsequent populations that remain close to their equilibrium values, then we say that the equilibrium point is **stable**. On the other hand, if a perturbation results in a radical change, with, for example, the population of foxes or rabbits increasing without limit, then we say that the equilibrium point is **unstable**.

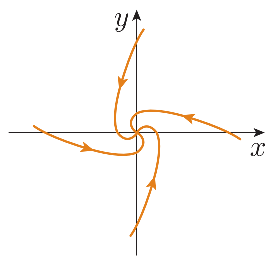


Figure 12 Phase paths close to a sink, which is a stable equilibrium point

If you look at the phase diagram in Figure 12, where the origin is a sink, you can see that any slight perturbation from the origin will result in a phase point that returns to the origin as time increases. So the point $(0, 0)$ is a *stable* equilibrium point.

On the other hand, the origin in the phase diagram shown in Figure 6 is an *unstable* equilibrium point. Any perturbation away from the origin will result in the phase point travelling further and further away from the origin with time. Similarly, the origin in the phase diagram shown in Figure 8 is an *unstable* equilibrium point. Apart from increases or decreases in y with x unchanged, any perturbation will result in a point that travels further and further away from the origin with time.

The stability of equilibrium points

Suppose that the system of differential equations

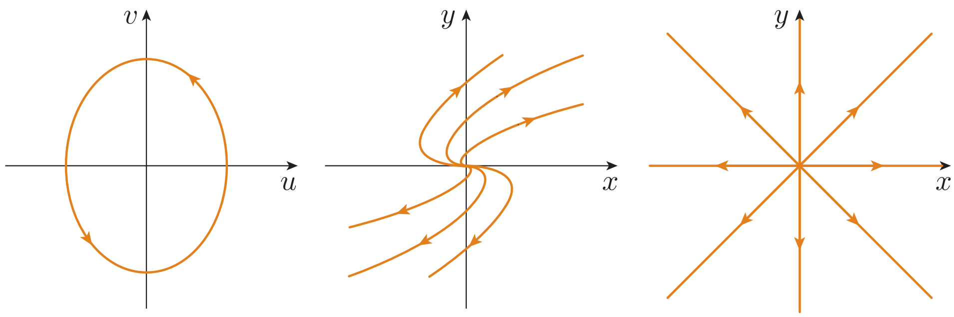
$$\dot{\mathbf{x}} = \mathbf{u}(x, y)$$

has an equilibrium point at $x = x_e$, $y = y_e$. The equilibrium point is said to be:

- **stable** if all points in the neighbourhood of the equilibrium point remain in the neighbourhood of the equilibrium point as time increases
- **unstable** otherwise.

Exercise 10

Classify the equilibrium points $(0, 0)$ shown in the following phase diagrams as stable or unstable.



(a) (b) (c)

2.3 Linearised equations of motion

In general, non-linear systems of differential equations are hard to solve, and it may be impossible to find exact solutions. However, we are often interested in situations where the system is close to an equilibrium point. In this case, it is sensible to approximate the non-linear equations by a suitable set of *linear* differential equations, which can be analysed by the methods of Unit 6.

We have another motive for finding linear approximations. In the next section you will see that there is a systematic way of investigating whether equilibrium points are stable or unstable. This method applies most directly to linear systems of equations. For a non-linear system, such as the Lotka–Volterra equations, a preliminary step is needed: we must first find linear approximations to the non-linear system that apply close to the equilibrium points. This subsection explains how this is done.

If (x_e, y_e) is an equilibrium point, consider small perturbations p and q from x_e and y_e, giving new values x and y defined by

$$x = x_e + p, \quad y = y_e + q. \tag{17}$$

Although a population x or y cannot be negative, a perturbation p or q can (usually) be negative if the population is less than the equilibrium value.

We can find the time development of the small perturbations p and q by **linearising** the differential equation $\dot{\mathbf{x}} = \mathbf{u}(x, y)$. We will make use of Taylor polynomials to achieve this.

In order to do so, we must write each component of the vector field $\mathbf{u}(x, y)$ as a function of the two variables x and y:

$$\mathbf{u}(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}.$$

At the equilibrium point (x_e, y_e), we have $\mathbf{u}(x_e, y_e) = \mathbf{0}$, i.e.

$$u(x_e, y_e) = 0 \quad \text{and} \quad v(x_e, y_e) = 0.$$

Now, for small perturbations p and q, we can use the linear Taylor polynomial for functions of two variables to approximate each of $u(x, y)$ and $v(x, y)$ near the equilibrium point (x_e, y_e). We have

$$\begin{aligned} u(x_e + p, y_e + q) &\simeq u(x_e, y_e) + p\,\frac{\partial u}{\partial x}(x_e, y_e) + q\,\frac{\partial u}{\partial y}(x_e, y_e) \\ &= p\,\frac{\partial u}{\partial x}(x_e, y_e) + q\,\frac{\partial u}{\partial y}(x_e, y_e), \end{aligned}$$

Note that $\frac{\partial u}{\partial x}(x_e, y_e)$ means the same thing as $\left.\frac{\partial u}{\partial x}\right|_{x=x_e,\ y=y_e}$, i.e. it is the partial derivative $\partial u/\partial x$ evaluated at the point $(x, y) = (x_e, y_e)$.

since $u(x_e, y_e) = 0$. Also,

$$\begin{aligned} v(x_e + p, y_e + q) &\simeq v(x_e, y_e) + p\,\frac{\partial v}{\partial x}(x_e, y_e) + q\,\frac{\partial v}{\partial y}(x_e, y_e) \\ &= p\,\frac{\partial v}{\partial x}(x_e, y_e) + q\,\frac{\partial v}{\partial y}(x_e, y_e), \end{aligned}$$

since $v(x_e, y_e) = 0$.

The above two equations may appear rather unwieldy, but are much more succinctly represented in matrix form:

$$\begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x}(x_e, y_e) & \frac{\partial u}{\partial y}(x_e, y_e) \\ \frac{\partial v}{\partial x}(x_e, y_e) & \frac{\partial v}{\partial y}(x_e, y_e) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

Since $x(t) = x_e + p(t)$ and $y(t) = y_e + q(t)$, we also have

$$\dot{x} = \dot{p}, \quad \dot{y} = \dot{q}.$$

Putting the pieces together, by substituting in $\dot{\mathbf{x}} = \mathbf{u}(x, y)$, gives a system of *linear* differential equations for the perturbations p and q:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x}(x_e, y_e) & \frac{\partial u}{\partial y}(x_e, y_e) \\ \frac{\partial v}{\partial x}(x_e, y_e) & \frac{\partial v}{\partial y}(x_e, y_e) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}. \tag{18}$$

An example will help to make this clear.

Example 3

For the Lotka–Volterra equations (15), determine the linearised equations that describe perturbations p and q from the equilibrium point (X, Y).

Solution

For the Lotka–Volterra equations we have

$$u(x, y) = kx\left(1 - \frac{y}{Y}\right),$$
$$v(x, y) = -hy\left(1 - \frac{x}{X}\right).$$

First we compute the partial derivatives, obtaining

$$\frac{\partial u}{\partial x} = k\left(1 - \frac{y}{Y}\right), \quad \frac{\partial u}{\partial y} = -\frac{kx}{Y},$$
$$\frac{\partial v}{\partial x} = \frac{hy}{X}, \quad \frac{\partial v}{\partial y} = -h\left(1 - \frac{x}{X}\right).$$

Evaluating these at the point (X, Y) gives

$$\frac{\partial u}{\partial x}(X, Y) = 0, \quad \frac{\partial u}{\partial y}(X, Y) = -\frac{kX}{Y},$$
$$\frac{\partial v}{\partial x}(X, Y) = \frac{hY}{X}, \quad \frac{\partial v}{\partial y}(X, Y) = 0.$$

Thus the required system of linear differential equations is

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & -kX/Y \\ hY/X & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}, \tag{19}$$

which can be written as the pair of equations

$$\dot{p} = -\frac{kX}{Y}q, \quad \dot{q} = \frac{hY}{X}p.$$

We have replaced a system of non-linear equations, for which we have no explicit solution, with a pair of *linear* equations that can be solved using methods introduced in Unit 6. We should expect the solutions of equation (19) to provide a good approximation to the original system when p and q are small (i.e. when the system is close to the equilibrium point (X, Y)).

The matrix

$$\mathbf{J}(x, y) = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

is called the **Jacobian matrix** of the vector field

$$\mathbf{u}(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}.$$

The 2×2 matrix on the right-hand side of equation (18) is this Jacobian matrix evaluated at the equilibrium point (x_e, y_e), so equation (18) can be written succinctly as

$$\dot{\mathbf{p}} = \mathbf{J}(x_\text{e}, y_\text{e})\,\mathbf{p},$$

where $\mathbf{p} = [p \quad q]^T$ is the perturbation from the equilibrium point (x_e, y_e).

Procedure 2 Linearising near an equilibrium point

Suppose that the system of differential equations

$$\dot{\mathbf{x}} = \mathbf{u}(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$$

has an equilibrium point at $x = x_\text{e}$, $y = y_\text{e}$.

1. Find the Jacobian matrix

$$\mathbf{J}(x, y) = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}.$$

2. In the neighbourhood of the equilibrium point (x_e, y_e), the differential equations can be approximated by the linearised form

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x}(x_\text{e}, y_\text{e}) & \dfrac{\partial u}{\partial y}(x_\text{e}, y_\text{e}) \\ \dfrac{\partial v}{\partial x}(x_\text{e}, y_\text{e}) & \dfrac{\partial v}{\partial y}(x_\text{e}, y_\text{e}) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix},$$

where $x(t) = x_\text{e} + p(t)$ and $y(t) = y_\text{e} + q(t)$.

The Jacobian matrix, evaluated at the equilibrium point, is the *matrix of coefficients* of the linearised system.

Exercise 11

Write down the linear approximations to the Lotka–Volterra equations (15) near the equilibrium point $(0, 0)$.

Exercise 12

Consider two populations modelled by the equations

$$\dot{x} = x(20 - y), \quad \dot{y} = y(10 - y)(10 - x) \quad (x \geq 0,\ y \geq 0).$$

These equations were considered in Exercise 9.

Find the linear approximations to these equations near the equilibrium point $(10, 20)$.

We have reduced the discussion of the behaviour of a pair of differential equations near an equilibrium point to an examination of the behaviour of a pair of *linear* differential equations. In the next section we will use the techniques from Unit 6 to solve these differential equations.

Exercise 13

Find the equilibrium point (x_e, y_e) of the system of differential equations

$$\dot{x} = 3x + 2y - 8,$$
$$\dot{y} = x + 4y - 6.$$

Find a system of linear differential equations satisfied by small perturbations p and q from the equilibrium point.

Exercise 14

Suppose that a pair of populations x and y can be modelled by the system of differential equations

$$\dot{x} = 0.5x - 0.000\,05x^2,$$
$$\dot{y} = -0.1y + 0.0004xy - 0.01y^2 \quad (x \geq 0,\ y \geq 0).$$

(a) Find the three equilibrium points of the system.

(b) Find the Jacobian matrix of the system.

(c) For each of the three equilibrium points, find the matrix form of the linear differential equations that give the approximate behaviour of the system near the equilibrium point.

3 Classifying equilibrium points

In the previous section you saw how a system of non-linear differential equations $\dot{\mathbf{x}} = \mathbf{u}(x, y)$ may be approximated near an equilibrium point by a linear system $\dot{\mathbf{p}} = \mathbf{A}\mathbf{p}$, where $\mathbf{A}$ is the Jacobian matrix

$$\mathbf{A} = \mathbf{J}(x_e, y_e).$$

In this section we develop an algebraic method of classification, based on the eigenvalues of this Jacobian matrix. The procedure for classifying an equilibrium point of a non-linear system will then be as follows.

Classifying equilibrium points

- Near an equilibrium point, approximate the non-linear system by a linear system.
- Find the eigenvalues of the Jacobian matrix for these linearised equations.
- Classify the equilibrium point of the linearised system using these eigenvalues.
- Deduce (where possible) the behaviour of the original system in the neighbourhood of the equilibrium point.

The linearised system of differential equations, which approximates the behaviour of the non-linear system in the neighbourhood of the equilibrium point, has the form

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix},$$

where p and q are perturbations from the equilibrium point, and a, b, c and d are constants. This is equivalent to the set of equations

The perturbations p and q can usually take negative, as well as positive, values.

$$\dot{p} = ap + bq,$$
$$\dot{q} = cp + dq.$$

From Unit 6, we know that the general solutions of such equations are determined by the eigenvalues and eigenvectors of the matrix of coefficients, which in the present context is the Jacobian matrix evaluated at the equilibrium point. We will illustrate various kinds of behaviour that can arise, by examining some examples. First, we look at matrices with real eigenvalues of various signs, then we consider matrices with complex eigenvalues. A summary of all these cases is given in Subsection 3.3.

3.1 Matrices with two real eigenvalues

Let us first consider the linear system of differential equations $\dot{\mathbf{p}} = \mathbf{A}\mathbf{p}$ where

For a linear system of differential equations, the Jacobian matrix is the matrix of coefficients, and the equilibrium point is at $p = 0$, $q = 0$.

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}. \tag{20}$$

(This system is very similar to the one considered in Example 1.) The matrix $\mathbf{A}$ is diagonal, so its eigenvalues are 2 and 3. The corresponding eigenvectors are $[1 \quad 0]^T$ and $[0 \quad 1]^T$, respectively. Following the method given in Unit 6, the general solution is constructed from these eigenvalues and eigenvectors, and is given by

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} = C \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + D \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t},$$

where C and D are arbitrary constants. Equivalently, we have

$$p(t) = Ce^{2t}, \quad q(t) = De^{3t}.$$

We are interested in the behaviour of phase paths near the equilibrium point at $p = 0$, $q = 0$. Consider, for example, the paths with $D = 0$ (and $C \neq 0$). On these paths we have $p(t) = Ce^{2t}$ and $q(t) = 0$, so the point $(p(t), q(t))$ moves away from the origin along the p-axis as t increases.

On the other hand, consider the paths with $C = 0$ (and $D \neq 0$). On these paths we have $p(t) = 0$ and $q(t) = De^{3t}$, so the point $(p(t), q(t))$ moves away from the origin along the q-axis as t increases. Hence we have seen that there are phase paths along the axes, corresponding to the eigenvectors $[1 \quad 0]^T$ and $[0 \quad 1]^T$. As t increases, a point on either of these axes moves away from the origin.

For general values of C and D, where neither $C = 0$ nor $D = 0$, the point (Ce^{2t}, De^{3t}) still moves away from the origin as t increases, but not along a straight line. On the other hand, as t decreases, the exponential functions e^{2t} and e^{3t} decrease (and tend to zero as t tends to $-\infty$). So the point (Ce^{2t}, De^{3t}) approaches the origin as t decreases, and all paths radiate from the origin. This is illustrated in Figure 13, where we have incorporated the fact that the only straight-line paths are the two axes, which correspond to the two eigenvectors of the matrix $\mathbf{A}$.

In this case, we can show that the paths are $q = K|p|^{3/2}$, as in Example 1, but in this section we are interested in the qualitative behaviour.

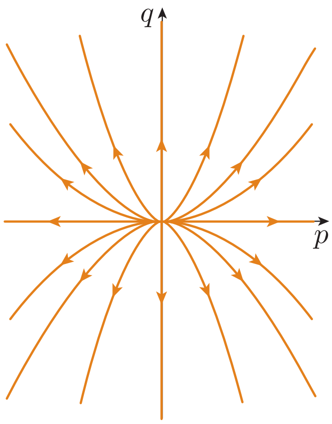


Figure 13 A source (unstable)

An equilibrium point with this type of qualitative behaviour in its neighbourhood is a **source**, and is unstable. This behaviour occurs for any linear system $\dot{\mathbf{p}} = \mathbf{A}\mathbf{p}$ where the Jacobian matrix $\mathbf{A}$ has *positive distinct eigenvalues*. The only straight-line paths are in the directions of the eigenvectors of the matrix $\mathbf{A}$, although these will not, in general, be along the axes.

Exercise 15

Consider the linear system of differential equations

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

(a) Find the eigenvalues of the Jacobian matrix.

(b) Classify the equilibrium point $p = 0$, $q = 0$ of the system.

Now consider the system of differential equations $\dot{\mathbf{p}} = \mathbf{A}\mathbf{p}$ where

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}. \tag{21}$$

The change in sign for matrix $\mathbf{A}$ from equation (20) to equation (21) changes the solution from one involving positive exponentials to one involving negative exponentials. You can think of this as replacing t by $-t$, so the solutions describe the same paths, but traversed in opposite directions. This changes the direction of the arrows along the paths in Figure 13.

So if the Jacobian matrix for a linear system has *negative distinct eigenvalues*, then the equilibrium point is a **sink**, and is stable. The only straight-line paths are along the directions of the eigenvectors of the Jacobian matrix.

Exercise 16

Consider the linear system of differential equations

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

(a) Find the eigenvalues of the Jacobian matrix.

(b) Classify the equilibrium point $p = 0$, $q = 0$ of the system.

These conclusions are modified slightly when the two eigenvalues happen to be equal, as the following exercise illustrates.

We include here the case where both eigenvalues are equal, but we will always assume that there are two linearly independent eigenvectors.

Exercise 17

Consider the linear system of differential equations

$$\begin{bmatrix}\dot{p}\\ \dot{q}\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}p\\ q\end{bmatrix},$$

which has eigenvalues 2 and 2, and eigenvectors $[1 \quad 0]^T$ and $[0 \quad 1]^T$.

(a) Find the general solution of the system of differential equations.

(b) By eliminating t, find the equations of the paths, and describe them.

(c) Is the equilibrium point $p = 0$, $q = 0$ stable or unstable?

In Exercise 17, you saw that when the Jacobian matrix has two real *identical positive eigenvalues* (but there are still *two linearly independent eigenvectors*), all the paths are straight lines radiating away from the origin, as shown in Figure 14. The equilibrium point at $p = 0$, $q = 0$ is then called a **star source** (and is unstable).

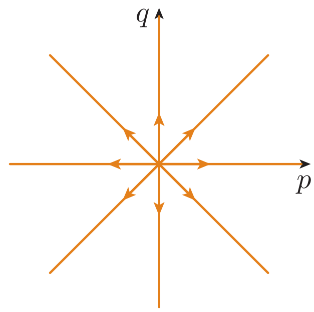


Figure 14 A star source (unstable)

If there are two real *identical negative eigenvalues* (but there are still *two linearly independent eigenvectors*), then the arrows on the paths in Figure 14 are reversed, and the equilibrium point at $p = 0$, $q = 0$ is called a **star sink**, which is stable. This is illustrated in Figure 15.

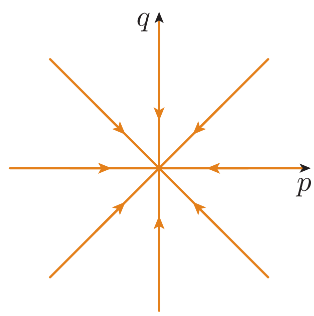


Figure 15 A star sink (stable)

So far in this section we have considered the case where the Jacobian matrix has two positive eigenvalues and the case where the matrix has two negative eigenvalues. We now consider the case where the matrix has *one positive eigenvalue and one negative eigenvalue.* For example, consider the matrix

$$\mathbf{A} = \begin{bmatrix}1 & 4\\ 1 & -2\end{bmatrix},$$

which has eigenvalues 2 and -3, and corresponding eigenvectors $[4 \quad 1]^T$ and $[1 \quad -1]^T$. The general solution of the linear system of differential equations $\dot{\mathbf{p}} = \mathbf{A}\mathbf{p}$ is therefore

$$\begin{bmatrix}p\\ q\end{bmatrix} = C\begin{bmatrix}4\\ 1\end{bmatrix}e^{2t} + D\begin{bmatrix}1\\ -1\end{bmatrix}e^{-3t}. \qquad (22)$$

When $D = 0$ (and $C \neq 0$), we have $p(t) = 4Ce^{2t}$ and $q(t) = Ce^{2t}$, and the point $(p(t), q(t))$ moves away from the origin along the straight-line path $q = \frac{1}{4}p$ as t increases. On the other hand, when $C = 0$ (and $D \neq 0$), the solution is $p(t) = De^{-3t}$, $q(t) = -De^{-3t}$, so the point $(p(t), q(t))$ approaches the origin along the straight-line path $q = -p$ as t increases.

Hence we have seen that there are two straight-line paths. On the line $q = \frac{1}{4}p$ (which corresponds to the eigenvector $[4 \quad 1]^T$), the point moves away from the origin as t increases. However, on the line $q = -p$ (which corresponds to the eigenvector $[1 \quad -1]^T$), the point moves towards the origin as t increases.

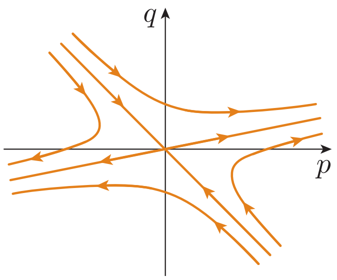


Figure 16 A saddle (unstable)

Now we will consider the behaviour of a general point $(p(t), q(t))$, where $p(t)$ and $q(t)$ are given by equation (22), and neither C nor D is zero. For large positive values of t, the terms involving e^{2t} dominate, so $p(t) \simeq 4Ce^{2t}$ and $q(t) \simeq Ce^{2t}$. So for large positive values of t, the general path approaches the line $q = \frac{1}{4}p$. On the other hand, for large negative values of t, the terms involving e^{-3t} dominate, so $p(t) \simeq De^{-3t}$ and $q(t) \simeq -De^{-3t}$. So for large negative values of t, the general path approaches the line $q = -p$. Using this information we can construct the paths in phase space, illustrated in Figure 16.

An equilibrium point with this type of behaviour is called a **saddle**. It occurs when the Jacobian matrix has *one positive eigenvalue and one negative eigenvalue*. Again, the two straight-line paths are in the directions of the eigenvectors of the matrix.

Exercise 18

Consider the linear system of differential equations

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

(a) Find the eigenvalues and corresponding eigenvectors of the Jacobian matrix.

(b) Classify the equilibrium point $p = 0$, $q = 0$.

(c) Sketch the phase paths of the solutions of the differential equations.

3.2 Matrices with complex eigenvalues

In Unit 5 you saw that some matrices with real matrix elements have *complex* eigenvalues and eigenvectors. However, in Unit 6 you saw that these complex quantities can be used to construct the *real* solutions of the corresponding system of linear differential equations. Our next example involves such a system.

Example 4

Consider the linear system of differential equations

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

(a) Find the eigenvalues and corresponding eigenvectors of the Jacobian matrix.

(b) Hence obtain the general solution of the system of differential equations.

(c) Show that the phase paths for these differential equations are the ellipses

$$p^2 + \tfrac{1}{4}q^2 = K,$$

where K is a positive constant.

The standard equation for an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are constants.

Solution

(a) The matrix has the characteristic equation

$$\begin{vmatrix} -\lambda & -1 \\ 4 & -\lambda \end{vmatrix} = 0,$$

i.e. $\lambda^2 + 4 = 0$. So the eigenvalues are $\lambda = 2i$ and $\lambda = -2i$.

When $\lambda = 2i$, the eigenvector $[a \quad b]^T$ satisfies the equation

$$\begin{bmatrix} -2i & -1 \\ 4 & -2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so $-2ia - b = 0$, which gives $b = -2ia$. Hence an eigenvector corresponding to the eigenvalue $\lambda = 2i$ is $[1 \quad -2i]^T$.

Similarly, an eigenvector corresponding to the eigenvalue $\lambda = -2i$ is $[1 \quad 2i]^T$.

(b) The general solution of the system can be found by the method explained in Unit 6. This involves finding the real and imaginary parts of $\mathbf{v}e^{\lambda t}$, where λ and $\mathbf{v}$ are an eigenvalue–eigenvector pair. Using the eigenvalue $2i$ and the corresponding eigenvector $[1 \quad -2i]^T$, we get

It does not matter which of the two eigenvalue–eigenvector pairs we choose, as they are complex conjugates of one another.

$$\begin{aligned} \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{2it} &= \begin{bmatrix} 1 \\ -2i \end{bmatrix} (\cos 2t + i \sin 2t) \\ &= \begin{bmatrix} \cos 2t \\ 2\sin 2t \end{bmatrix} + i \begin{bmatrix} \sin 2t \\ -2\cos 2t \end{bmatrix}. \end{aligned}$$

The general solution can then be written down as

See Procedure 2 and Example 5 of Unit 6.

$$\begin{bmatrix} p \\ q \end{bmatrix} = C \begin{bmatrix} \cos 2t \\ 2\sin 2t \end{bmatrix} + D \begin{bmatrix} \sin 2t \\ -2\cos 2t \end{bmatrix},$$

where C and D are arbitrary constants.

(c) We have

$$\begin{aligned} p(t) &= C\cos 2t + D\sin 2t, \\ q(t) &= 2C\sin 2t - 2D\cos 2t, \end{aligned}$$

so

$$\begin{aligned} p^2 + \tfrac{1}{4}q^2 &= (C\cos 2t + D\sin 2t)^2 + (C\sin 2t - D\cos 2t)^2 \\ &= (C^2\cos^2 2t + 2CD\cos 2t\sin 2t + D^2\sin^2 2t) \\ &\quad + (C^2\sin^2 2t - 2CD\cos 2t\sin 2t + D^2\cos^2 2t) \\ &= C^2(\cos^2 2t + \sin^2 2t) + D^2(\cos^2 2t + \sin^2 2t) \\ &= C^2 + D^2 = K, \end{aligned}$$

where $K = C^2 + D^2$.

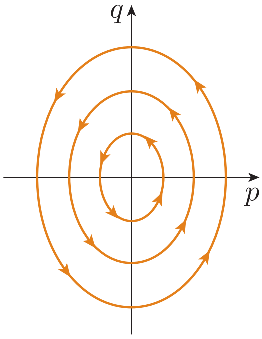


Figure 17 A centre (stable)

So the phase paths are ellipses, as shown in Figure 17. The direction of the arrows can be deduced from the original differential equations. For example, in the first quadrant, $\dot{p} = -q < 0$ and $\dot{q} = 4p > 0$.

In Example 4, you saw that the phase paths are ellipses. This type of behaviour corresponds to any linear system of differential equations where the eigenvalues of the Jacobian matrix are *purely imaginary*. An equilibrium point that has this behaviour in its neighbourhood is called a *centre*, and is stable. More generally, if all the phase paths in the vicinity of an equilibrium point are *closed curves*, then the equilibrium point is stable and is called a **centre**.

Exercise 19

Consider the linear system of differential equations

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

(a) Find the eigenvalues of the Jacobian matrix.

(b) Classify the equilibrium point $p = 0$, $q = 0$.

In general, when the eigenvalues of a matrix are complex, they are not purely imaginary but also contain a real part. This has a significant effect on the solution of the corresponding system, as you will see in the following example.

Example 5

Find the general solution of the system of equations $\dot{\mathbf{p}} = \mathbf{A}\mathbf{p}$, where

$$\mathbf{A} = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}.$$

Sketch some phase paths corresponding to the solutions of the system.

Solution

The characteristic equation of the Jacobian matrix is $(2+\lambda)^2 + 9 = 0$, so the eigenvalues are $-2+3i$ and $-2-3i$. The corresponding eigenvectors are $[1 \quad -i]^T$ and $[1 \quad i]^T$, respectively. To construct the general solution, we need to find the real and imaginary parts of $[1 \quad -i]^T\, e^{(-2+3i)t}$. We get

$$\begin{aligned} e^{-2t}e^{3it} \begin{bmatrix} 1 \\ -i \end{bmatrix} &= e^{-2t}(\cos 3t + i \sin 3t) \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &= e^{-2t} \begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + ie^{-2t} \begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix}. \end{aligned}$$

So the general solution is

$$\begin{bmatrix} p \\ q \end{bmatrix} = Ce^{-2t} \begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + De^{-2t} \begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix},$$

where C and D are arbitrary constants.

If we neglect, for the time being, the e^{-2t} factors, the solution is

$$p = C\cos 3t + D\sin 3t,$$
$$q = C\sin 3t - D\cos 3t,$$

from which it follows that

$$p^2 + q^2 = C^2 + D^2$$

(the argument is similar to that given in the solution to Example 4).

So, in the absence of the e^{-2t} terms, the paths would be circles centred on the origin. The effect of the e^{-2t} terms on these paths is to reduce the radius of the circles gradually. In other words, the paths spiral in towards the origin as t increases, as shown in Figure 18.

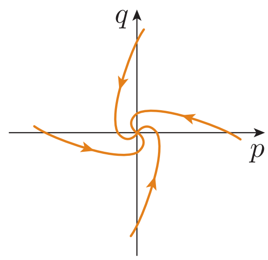


Figure 18 A spiral sink (stable)

In Example 5 the paths spiral in towards the origin, so the origin is a sink (which is called a **spiral sink**) and therefore is a *stable* equilibrium point. If the paths spiralled away from the origin, we would have a **spiral source** (Figure 19) with the equilibrium point *unstable*. The stability is determined by the sign of the real part of the complex eigenvalues. If the real part is positive, then the general solution involves e^{kt} terms (where k is positive) and the equilibrium point is an unstable spiral source; if the real part is negative, then the general solution involves e^{-kt} terms and the equilibrium point is a stable spiral sink.

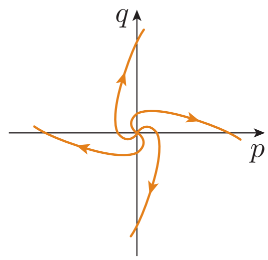


Figure 19 A spiral source (unstable)

Exercise 20

Consider the linear system of differential equations

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

(a) Find the eigenvalues of the Jacobian matrix.

(b) Classify the equilibrium point $p = 0$, $q = 0$.

3.3 Classifying equilibrium points of linear systems

We now summarise the results of the previous three subsections. The phase diagrams are collected together in Figure 20.

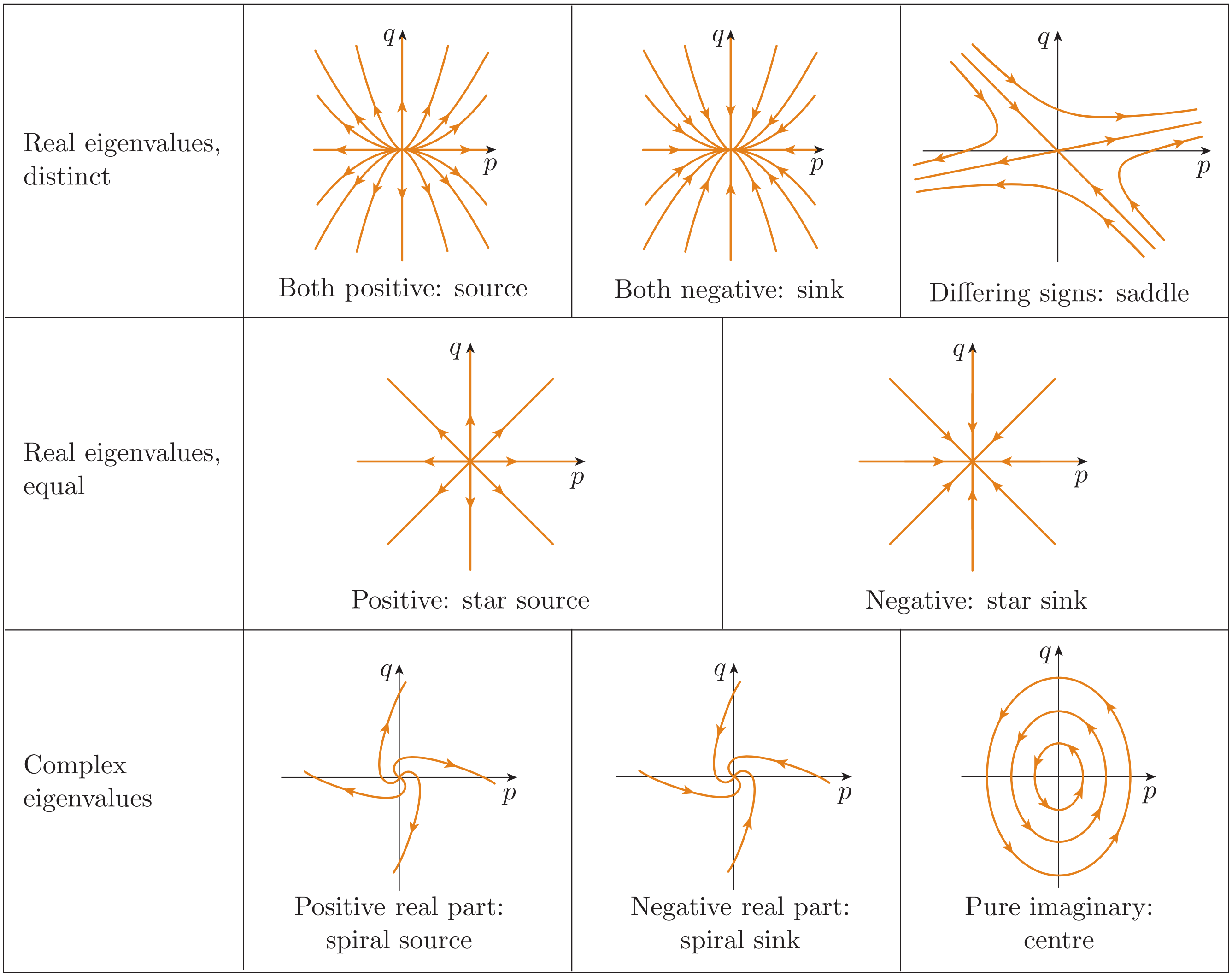


Figure 20 Phase diagrams of linearised equations

Procedure 3 Classification of the equilibrium points of a linear system

Consider the linear system $\dot{\mathbf{p}} = \mathbf{A}\mathbf{p}$, for a 2×2 matrix $\mathbf{A}$. The nature of the equilibrium point at $p = 0$, $q = 0$ is determined by the eigenvalues and eigenvectors of $\mathbf{A}$.

1. If the eigenvalues are **real and distinct**, then:
   - if both eigenvalues are positive, the equilibrium point is a *source* (and is unstable)
   - if both eigenvalues are negative, the equilibrium point is a *sink* (and is stable)
   - if one of the eigenvalues is positive and the other is negative, the equilibrium point is a *saddle* (and is unstable).

2. If the eigenvalues are **real and equal** (and there are *two linearly independent eigenvectors*), then:
   - if the eigenvalues are positive, the equilibrium point is a *star source* (and is unstable)
   - if the eigenvalues are negative, the equilibrium point is a *star sink* (and is stable).
3. If the eigenvalues are **complex**, then:
   - if the eigenvalues are purely imaginary, the equilibrium point is a *centre* (and is stable)
   - if the eigenvalues have a positive real part, the equilibrium point is a *spiral source* (and is unstable)
   - if the eigenvalues have a negative real part, the equilibrium point is a *spiral sink* (and is stable).

Procedure 3 is not exhaustive; for example, it does not include a number of special cases, such as where one of the eigenvalues is zero.

Exercise 21

(a) Suppose that all you are told about a given equilibrium point is that both the eigenvalues of its Jacobian matrix are positive, or have a positive real part. What are the possible types of equilibrium point that fit this description?

(b) Suppose that all you are told about a given equilibrium point is that both the eigenvalues of its Jacobian matrix are negative, or have a negative real part. What are the possible types of equilibrium point that fit this description?

Exercise 22

In Example 3 you saw that the Lotka–Volterra equations can be approximated by the system of linear differential equations

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & -kX/Y \\ hY/X & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \qquad (k > 0 \text{ and } h > 0)$$

in the neighbourhood of the equilibrium point (X, Y). Find the eigenvalues of the Jacobian matrix, and hence classify the equilibrium point $p = 0$, $q = 0$ for the linearised system around (X, Y).

Exercise 23

In Exercise 11 you saw that the Lotka–Volterra equations can be approximated by the system of linear differential equations

$$\begin{bmatrix}\dot{p}\\ \dot{q}\end{bmatrix} = \begin{bmatrix}k & 0\\ 0 & -h\end{bmatrix}\begin{bmatrix}p\\ q\end{bmatrix}$$

in the neighbourhood of the equilibrium point $(0,0)$. Find the eigenvalues of the Jacobian matrix, and hence classify the equilibrium point $p=0$, $q=0$ for the linearised system around $(0,0)$.

3.4 Classifying equilibrium points of non-linear systems

In Section 2 you saw how to find the equilibrium points of non-linear systems of differential equations $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x})$, and how to find the linear system $\dot{\mathbf{p}} = \mathbf{A}\mathbf{p}$ that approximates the system in the neighbourhood of an equilibrium point. In this section you have seen how to classify an equilibrium point of the linear system by finding the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. But is the behaviour of the non-linear system near the equilibrium point the same as the behaviour of the linear system that approximates it? It can be shown that, not surprisingly, the answer is yes, *except* when the equilibrium point of the approximating linear system is a centre.

Near a centre, the paths of the approximating linear system are circular or elliptical. However, the paths of the original non-linear system may spiral towards or away from the equilibrium point, or they may be closed curves. In such cases, we can say only that the paths are approximately circular or elliptical; we can say nothing about their actual behaviour without further examination. Thus, if the linear approximation has a centre, we cannot immediately deduce the nature of the equilibrium point of the original non-linear system: it may be a stable centre, a stable spiral sink or an unstable spiral source.

Procedure 4 Classification of the equilibrium points of a non-linear system

To classify the equilibrium points of the non-linear system of differential equations

$$\dot{\mathbf{x}} = \mathbf{u}(x,y),$$

do the following.

1. Find the equilibrium points by using Procedure 1.
2. Use Procedure 2 to find the linear system

   $$\dot{\mathbf{p}} = \mathbf{A}\mathbf{p}$$

   that approximates the original non-linear system in the neighbourhood of each equilibrium point.
3. For each equilibrium point, use Procedure 3 to classify the linear system.

The behaviour of the original non-linear system near an equilibrium point is the same as that of the linear approximation, except when the linear system has a centre. If the linear system has a centre, the equilibrium point of the original non-linear system may be a centre (stable), a spiral sink (stable) or a spiral source (unstable).

In the case of the Lotka–Volterra equations (15), in Exercise 23 you have seen that the linear system of differential equations that approximates the non-linear system in the neighbourhood of the equilibrium point $(0,0)$ has an (unstable) *saddle* at the equilibrium point. So the (non-linear) Lotka–Volterra equations also have an (unstable) *saddle* at the equilibrium point $(0,0)$.

You also saw in Exercise 22 that the linear system of differential equations that approximates the Lotka–Volterra equations in the neighbourhood of the equilibrium point (X,Y) has a (stable) *centre* at the equilibrium point. This means that we cannot immediately say anything about the classification of this equilibrium point of the original (non-linear) system of differential equations – it could be a centre, a spiral sink or a spiral source. However, further investigation (using a method that will be introduced in Subsection 3.5) shows that with the exception of the equilibrium points and the coordinate axes, *every* phase path of the Lotka–Volterra equations is a closed path. (These closed paths are not ellipses, however.) So the equilibrium point (X,Y) of the Lotka–Volterra equations *is* a (stable) *centre*, as shown in Figure 21.

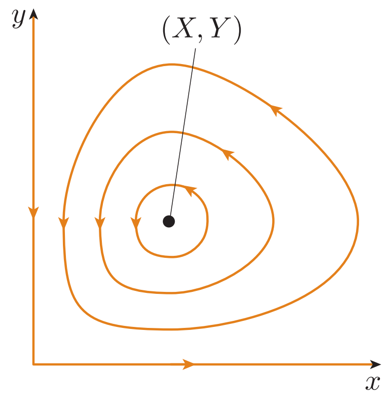


Figure 21 Phase diagram for the Lotka–Volterra equations, showing closed phase paths and a centre at (X,Y)

The following important example illustrates the steps involved in locating and classifying equilibrium points.

Example 6

Consider the non-linear system of differential equations

$$\begin{aligned} \dot{x} &= -4y + 2xy - 8, \\ \dot{y} &= 4y^2 - x^2. \end{aligned}$$

(a) Find the equilibrium points of the system.

(b) Compute the Jacobian matrix of the system.

(c) In the neighbourhood of each equilibrium point:

- linearise the system of differential equations
- classify the equilibrium point of the linearised system.

(d) What can you say about the classification of the equilibrium points of the original (non-linear) system of differential equations?

Solution

(a) The equilibrium points are given by

$$-4y + 2xy - 8 = 0,$$
$$4y^2 - x^2 = 0.$$

The second equation gives

$$x = \pm 2y.$$

When $x = 2y$, substitution into the first equation gives

$$-4y + 4y^2 - 8 = 0,$$

or $y^2 - y - 2 = 0$, which factorises to give

$$(y - 2)(y + 1) = 0.$$

Hence

$$y = 2 \quad \text{or} \quad y = -1.$$

When $y = 2$, $x = 2y = 4$. When $y = -1$, $x = 2y = -2$. So we have found two equilibrium points, namely $(4, 2)$ and $(-2, -1)$.

When $x = -2y$, substitution into the first equation gives

$$-4y - 4y^2 - 8 = 0,$$

or $y^2 + y + 2 = 0$. This quadratic equation has no real solutions, so there are no more equilibrium points.

(b) With the usual notation,

$$u(x, y) = -4y + 2xy - 8,$$
$$v(x, y) = 4y^2 - x^2.$$

So the Jacobian matrix is

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 2y & 2x - 4 \\ -2x & 8y \end{bmatrix}.$$

(c) At the equilibrium point $(4, 2)$, the Jacobian matrix is

$$\begin{bmatrix} 4 & 4 \\ -8 & 16 \end{bmatrix}.$$

The characteristic equation of this Jacobian matrix is

$$(4 - \lambda)(16 - \lambda) + 32 = 0,$$

or $\lambda^2 - 20\lambda + 96 = 0$, which factorises to give

$$(\lambda - 8)(\lambda - 12) = 0,$$

so the eigenvalues are

$$\lambda = 8 \quad \text{and} \quad \lambda = 12.$$

The two eigenvalues are positive and distinct, so the equilibrium point $p = 0$, $q = 0$ is a *source*, which is unstable.

At the equilibrium point $(-2, -1)$, the Jacobian matrix is

$$\begin{bmatrix} -2 & -8 \\ 4 & -8 \end{bmatrix}.$$

The characteristic equation of this Jacobian matrix is

$$(-2 - \lambda)(-8 - \lambda) + 32 = 0,$$

which simplifies to

$$\lambda^2 + 10\lambda + 48 = 0.$$

The roots of this quadratic equation are

$$\lambda = \frac{-10 \pm \sqrt{100 - 192}}{2} = -5 \pm i\sqrt{23},$$

so the eigenvalues are complex with a negative real part. Hence the equilibrium point $p = 0$, $q = 0$ is a *spiral sink*, which is stable.

(d) As neither of the equilibrium points found in part (c) is a centre, the non-linear system has an equilibrium point $(4, 2)$ that is a source (unstable), and an equilibrium point $(-2, -1)$ that is a spiral sink (stable).

Exercise 24

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix},$$

and hence find the general solution of the system $\dot{\mathbf{p}} = \mathbf{A}\mathbf{p}$. Classify the equilibrium point $p = 0$, $q = 0$.

If this system is the linear approximation to a non-linear system $\dot{\mathbf{x}} = \mathbf{u}(x, y)$ in the neighbourhood of an equilibrium point, what can you say about this equilibrium point of the non-linear system?

Exercise 25

Consider the non-linear system of differential equations

$$\dot{x} = (1 + x - 2y)x,$$
$$\dot{y} = (x - 1)y.$$

(a) Find the equilibrium points of the system.

(b) Find the Jacobian matrix of the system.

(c) In the neighbourhood of each equilibrium point:

- find the linear system of differential equations that gives the approximate behaviour of the system near the equilibrium point
- find the eigenvalues of the Jacobian matrix
- use the eigenvalues to classify the equilibrium point of the linearised system.

(d) What can you say about the classification of the equilibrium points of the original non-linear system of differential equations?

3.5 Constants of motion

You saw in Exercise 22 that the (non-linear) Lotka–Volterra equations can be linearised about the equilibrium point (X, Y), and that this point is a (stable) centre for the linearised system. However, this is not enough to show that (X, Y) is a stable equilibrium point of the original non-linear equations. In fact, (X, Y) *is* a stable centre of the Lotka–Volterra equations, as shown in Figure 21, but how can we establish this fact?

The trick is to show that there is a **constant of motion**. This is a function $K(x, y)$ that remains constant as we follow any given phase path. To see what this means, consider a function $K(x, y)$ as x and y trace out a path in phase space. Then the rate of change of $K(x, y)$ with respect to t is obtained by applying the chain rule of Unit 7. We have

$$\frac{dK}{dt} = \frac{\partial K}{\partial x}\frac{dx}{dt} + \frac{\partial K}{\partial y}\frac{dy}{dt}, \tag{23}$$

where $dx/dt = \dot{x}$ and $dy/dt = \dot{y}$ are the components of the velocity of the phase point in phase space: in this context, these are given by the Lotka–Volterra equations (15). The function $K(x, y)$ is a constant of motion if $dK/dt = 0$, and this can be tested by checking that the right-hand side of equation (23) vanishes for all x and y.

If we can find a function $K(x, y)$ that is a constant of motion for a given set of equations, it follows that K remains constant as we trace out any given phase path. This means that the phase paths are coincident with the contour lines of $K(x, y)$, which can be investigated quite easily.

Even if a constant of motion exists for a given system of differential equations, finding the appropriate function $K(x,y)$ is not easy and often requires informed guesswork. You will not be asked to do this, but you may be asked to verify that a given function is a constant of motion, using equation (23).

In the case of the Lotka–Volterra equations, it turns out that there is a constant of motion, namely

$$K(x,y) = h\ln x + k\ln y - \frac{h}{X}x - \frac{k}{Y}y, \tag{24}$$

where h and k are positive constants. The contour lines of this scalar field are plotted in Figure 22; they are closed curves and have the same shape as the phase paths plotted earlier, in Figure 21. Depending on the initial conditions, the phase point orbits around a particular contour, so that the motion of the Lotka–Volterra model is always periodic in time. The period depends on which contour is followed.

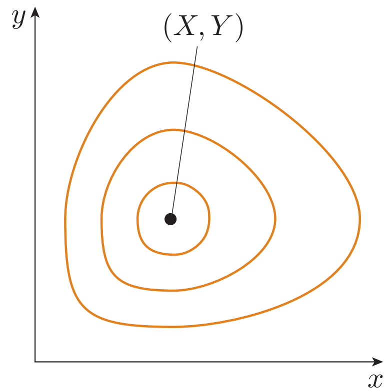


Figure 22 Contour lines of the scalar field $K(x,y)$ for the Lotka–Volterra equations

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Example 7

Show that

$$K(x,y) = h\ln x + k\ln y - \frac{h}{X}x - \frac{k}{Y}y$$

is a constant of motion for the Lotka–Volterra equations.

Solution

The partial derivatives are

$$\frac{\partial K}{\partial x} = \frac{h}{x} - \frac{h}{X} = \frac{h(X-x)}{xX},$$
$$\frac{\partial K}{\partial y} = \frac{k}{y} - \frac{k}{Y} = \frac{k(Y-y)}{yY}.$$

From the Lotka–Volterra equations (15), the velocity components are

$$\dot{x} = \frac{kx(Y-y)}{Y} \quad \text{and} \quad \dot{y} = \frac{-hy(X-x)}{X}.$$

The rate of change of K is therefore

$$\frac{dK}{dt} = \frac{hk}{XY}\big[(X-x)(Y-y) - (Y-y)(X-x)\big] = 0.$$

A more general understanding of phase paths

We have addressed behaviour in the immediate vicinity of an equilibrium point. But what happens more generally?

If there is a constant of motion, then the path follows its contours: these could be closed curves (as for the Lotka–Volterra equations), or they could be open curves, which run off to infinity.

But having a constant of motion is a special case; for 'most' choices of the functions $u(x, y)$ and $v(x, y)$ in equations (1), there is no constant of motion. In order to appreciate what can happen in the general case, we need to describe a behaviour that we have not yet encountered. This is where paths converge towards a single closed curve, which is called a **limit cycle**.

Figure 23 illustrates the distinction between the spiralling paths around a limit cycle (part (a)) and the closed paths that arise when there is a constant of motion (part (b)). Around a limit cycle, paths corresponding to different initial conditions all approach the *same* closed curve (the limit cycle) as $t \to \infty$. When there is a constant of motion with closed contour lines, we have a set of *distinct* closed paths, each characterised by different initial conditions.

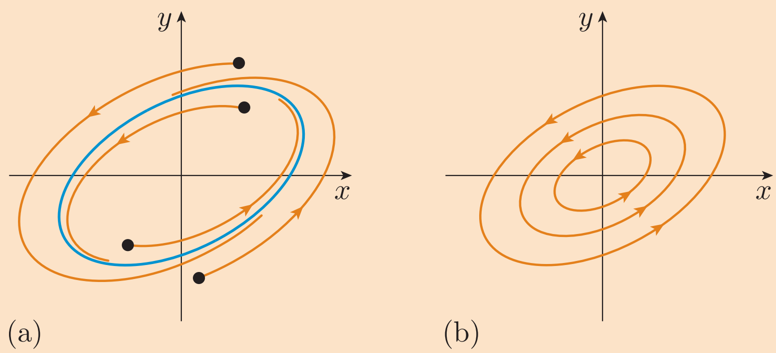


Figure 23 Contrasting phase paths around (a) a limit cycle (in blue) and (b) a centre

So what happens in the typical case, where there is no constant of motion? The first systematic study was made in the 1880s by Henri Poincaré (Figure 24), who showed that most two-dimensional, autonomous systems have paths that do one of three things:

- approach a stable equilibrium point
- approach a limit cycle
- run off to infinity.

The chaotic behaviour illustrated in Figure 1 is not consistent with the path approaching a limit cycle. Poincaré appreciated that chaotic motion could exist, but that it requires at least three coupled equations (in the case of autonomous systems).

Figure 24 Henri Poincaré (1854–1912) and his flamboyant signature. He pioneered the geometric analysis of systems of differential equations, along with important contributions to many other topics in mathematics, physics and engineering.

4 Motion of a rigid pendulum

In the Introduction we mentioned that non-linear differential equations arise in the description of mechanical systems. We will present a brief discussion of just one mechanical system, taking its equation of motion as given. This is included because it illustrates a useful general principle: higher-order differential equations are often written as systems of first-order equations in a larger number of variables. There are various advantages in this approach. Here we emphasise that it leads to a graphical representation of the motion in phase space. This helps us to give qualitative descriptions of differential equations that we cannot solve exactly.

The derivation of equations of motion often requires special techniques, such as those of *Lagrangian mechanics*, and is beyond the scope of this module.

4.1 Equations of motion for a rigid pendulum

We consider the motion of a rigid pendulum, illustrated in Figure 25. There is a mass m at the end of a rigid rod, which moves freely in a fixed vertical plane. Let θ (measured in radians) be the angular displacement from the downward vertical in an anticlockwise direction. When frictional forces can be neglected, the equation of motion for θ as a function of time t is the **rigid pendulum equation** or **undamped pendulum equation**

$$\ddot{\theta} = -\omega^2 \sin\theta, \tag{25}$$

where ω is a positive constant. You can think of this as a form of Newton's second law where the force is proportional to $\sin\theta$, but a satisfactory derivation requires a relatively sophisticated approach. Note that the force is zero when $\theta = \pi$, as well as when $\theta = 0$. This is a consequence of the fact that the rigid pendulum can, in principle, be balanced so that the mass is vertically above the pivot.

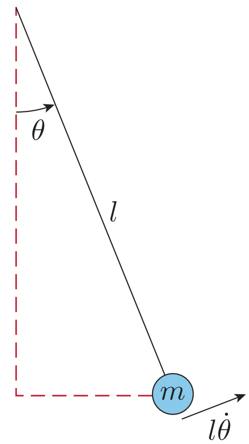


Figure 25 A rigid pendulum consisting of a mass (sometimes called a bob) at the end of a pivoted rigid rod

We also mention two closely-related equations.

If the displacement of the pendulum is small, you can use the approximation $\sin\theta \simeq \theta$, and the equation of motion (25) is replaced by an equation that we refer to as the **simple pendulum equation**

$$\ddot{\theta} = -\omega^2\theta. \tag{26}$$

This equation has the advantage that it is linear, but the disadvantage that it is a good approximation only for small oscillations. Of course, it is the same as the equation for simple harmonic motion, which was discussed in Unit 3.

We also consider the **damped pendulum equation**

$$\ddot{\theta} = -\omega^2 \sin\theta - \varepsilon\dot{\theta}, \tag{27}$$

where ε is a positive constant. This equation is very similar to the equation for the damped harmonic oscillator, also considered in Unit 3: the term proportional to $\dot{\theta} = d\theta/dt$ represents the effect of a frictional force that resists the movement of the rod.

4.2 The phase plane for a pendulum

In the Introduction we mentioned that the second-order differential equation (25) can be written as two first-order equations. Here we return to look at this point in greater depth.

Although the differential equations (25), (26) and (27) in the previous subsection are of second order, we can rewrite each of them as a *pair* of first-order differential equations; this will enable us to use the techniques from earlier in the unit. More precisely, we will replace θ by x and $\dot{\theta}$ by y, so that

$$y = \dot{x} = \dot{\theta}$$

and

$$\dot{y} = \ddot{x} = \ddot{\theta}.$$

So, for example, equation (25) can be rewritten as the system of first-order differential equations

$$\dot{x} = y,$$
$$\dot{y} = -\omega^2 \sin x,$$

which can be rewritten in terms of a vector field as

$$\dot{\mathbf{x}} = \mathbf{u}(x, y), \tag{28}$$

where

$$\mathbf{u}(x, y) = \begin{bmatrix} y \\ -\omega^2 \sin x \end{bmatrix}.$$

Let us consider the significance of equation (28). Starting from (25), which is a second-order differential equation in a single variable θ, we transformed this into (28), which is a first-order differential equation describing the motion of a point in a two-dimensional phase space, where the coordinates are the angle $x = \theta$ and its rate of change $y = \dot{\theta}$.

Exercise 26

Using the technique employed above, rewrite equation (27) as a system of first-order differential equations.

So the two models introduced in Subsection 4.1 for the motion of a pendulum when the oscillations can be large give rise to two pairs of first-order differential equations.

The pendulum equations

- For arbitrarily large oscillations and no friction, we have

$$\dot{x} = y, \quad \dot{y} = -\omega^2 \sin x, \tag{29}$$

arising from the *undamped pendulum equation*.

- For arbitrarily large oscillations and a frictional force, we have

$$\dot{x} = y, \quad \dot{y} = -\omega^2 \sin x - \varepsilon y, \tag{30}$$

arising from the *damped pendulum equation.*

The analogy with our previous discussion of two interacting populations should be immediately obvious, but here the variables x and y are even more closely related than before, since one is the derivative of the other. A phase point representing a solution of equations (29) or (30) at a given time would tell us not only the position of the pendulum bob, but also its velocity.

For a pendulum, the variable $x = \theta$ represents an angle measured in radians, so the points $(x + 2\pi, y)$ and (x, y) represent the same state of the system. We could restrict the range of x to $-\pi < x \leq \pi$, although we could use any interval of length 2π, such as $0 \leq x < 2\pi$, for example.

Although we can solve the simple pendulum equation (26), we cannot find simple analytical solutions of the undamped and damped pendulum equations (25) and (27). However, we can use the techniques developed in Sections 1–3 of this unit to investigate the qualitative behaviour of the solutions of these equations.

Exercise 27

(a) Find the equilibrium points of the system described by equations (29), i.e.

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -\omega^2 \sin x \quad (-\pi < x \leq \pi), \end{aligned}$$

which is the system of differential equations arising from the undamped pendulum equation.

(b) Describe physically the two equilibrium points that you found in part (a). On physical grounds, would you expect these equilibrium points to be stable or unstable?

Exercise 28

(a) Find the equilibrium points of the system described by equations (30), i.e.

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -\omega^2 \sin x - \varepsilon y \quad (-\pi < x \leq \pi), \end{aligned}$$

which is the system of differential equations arising from the damped pendulum equation.

(b) Describe physically the two equilibrium points that you found in part (a). On physical grounds, would you expect these equilibrium points to be stable or unstable?

In Exercise 27, you showed that the undamped pendulum has two equilibrium points. The first equilibrium point is the origin $x = 0$, $y = 0$, which corresponds to the pendulum hanging vertically downwards at rest, and physically we expect this to be stable. The second equilibrium point is $x = \pi$, $y = 0$, which corresponds to a stationary pendulum pointing vertically upwards, which we would not expect to be stable. To classify these equilibrium points mathematically we must first consider the corresponding linearised equations.

Exercise 29

Consider the non-linear system of differential equations

$$\dot{x} = y,$$
$$\dot{y} = -\omega^2 \sin x,$$

which is the system of differential equations arising from the undamped pendulum equation.

(a) Find the Jacobian matrix of the system.

(b) In the neighbourhood of each of the equilibrium points $(0, 0)$ and $(\pi, 0)$:

- find the linear system of differential equations that gives the approximate behaviour of the non-linear system near the equilibrium point
- find the eigenvalues of the Jacobian matrix
- use the eigenvalues to classify the equilibrium point of the linearised system.

We have seen that an undamped rigid pendulum has two equilibrium points, at $x = \pi$, $y = 0$ (corresponding to a stationary pendulum pointing vertically upwards) and at $x = 0$, $y = 0$ (corresponding to a stationary pendulum hanging vertically downwards. In the approximation of linearised equations, the point at $x = \pi$, $y = 0$ is a saddle (which is unstable) and the point at $x = 0$, $y = 0$ is a centre (which is stable).

For the non-linear rigid pendulum equations, we can again conclude that the equilibrium point at $x = \pi$, $y = 0$ is a saddle (which is unstable). However, the equilibrium point at $x = 0$, $y = 0$ may be either a centre (stable), or a spiral sink (stable) or a spiral source (unstable). This uncertainty can be resolved by noting that the non-linear rigid pendulum equations have a constant of motion. An argument similar to that given in Subsection 3.5 can then be used to show that the $x = 0$, $y = 0$ equilibrium point is a centre and is therefore stable. This makes good sense physically: the pendulum undergoes periodic motion as it swings to and fro.

Exercise 30

Use the equations of motion for an undamped pendulum, equations (29), to show that

$$E(x,y) = \tfrac{1}{2}y^2 + \omega^2(1 - \cos x)$$

is a constant of motion for the undamped pendulum. (If you have studied mechanics, you may recognise that this is proportional to the sum of the kinetic energy and the gravitational potential energy.)

So far we have investigated the motion of the pendulum in the neighbourhood of the equilibrium points. But what about the motions that are not close to the equilibrium points? We can investigate these by considering the *vector field*, which is shown in Figure 26 along with the associated phase paths. Two of the paths shown represent the pendulum continuously circling the support in the same direction, with the value of x $(= \theta)$ always increasing or decreasing, so that the bob passes repeatedly through the vertical. The path $EFGH$ describes an anticlockwise rotation of the bob, and $IJKL$ describes a clockwise rotation. These paths do not look closed on the diagram, but they really are because points for which x differs by 2π are equivalent.

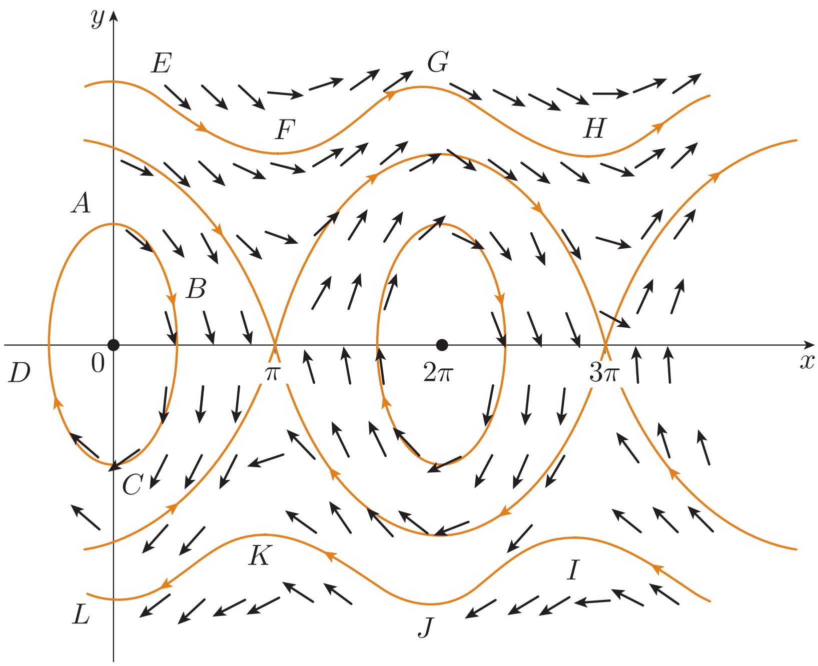


Figure 26 Phase diagram for a rigid pendulum; points for which x differs by 2π are equivalent

Figure 26 also shows a path that is a closed curve $ABCD$: this corresponds to a swinging motion of the pendulum, where there is a maximum angle of deflection and where the sign of the angular velocity can be positive or negative. Finally, the figure also shows a special phase curve where the pendulum reaches $x = \pi$ and $x = 3\pi$ with zero angular velocity ($y = \dot{\theta} = 0$). This curve, which divides the rotational and vibrational motions, is called the *separatrix*.

In the following exercises, we investigate the behaviour of the damped pendulum.

Exercise 31

Consider the non-linear system of differential equations

$$\dot{x} = y,$$
$$\dot{y} = -\omega^2 \sin x - \varepsilon y,$$

arising from the damped pendulum equation.

(a) Find the Jacobian matrix of the system.

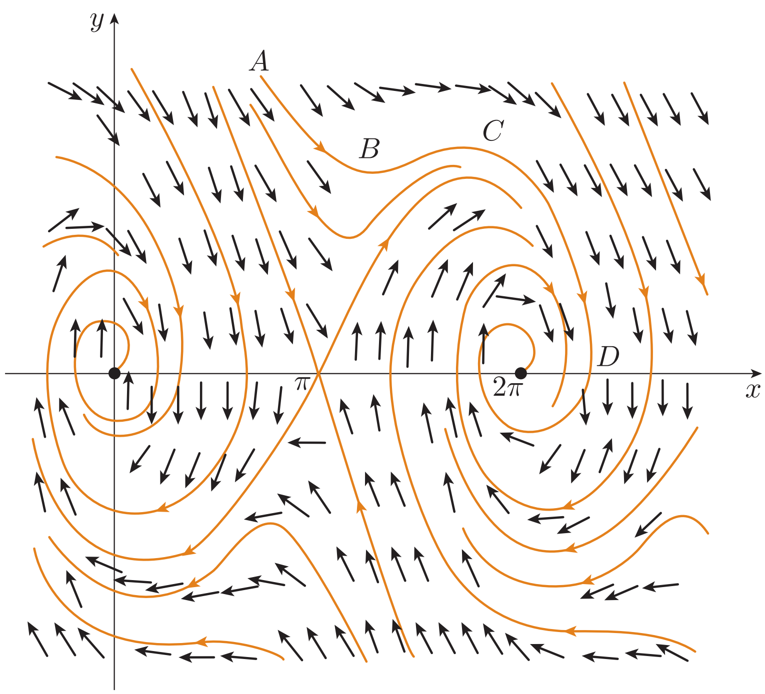
(b) In the neighbourhood of each of the equilibrium points $(0, 0)$ and $(\pi, 0)$:

- find the linear system of differential equations that gives the approximate behaviour of the non-linear system near the equilibrium point
- find the eigenvalues of the Jacobian matrix
- use the eigenvalues to classify the equilibrium point of the linearised system.

Note: For the equilibrium point $(0, 0)$, you will need to treat the cases $0 < \varepsilon < 2\omega$ and $\varepsilon > 2\omega$ separately. You may ignore the possibility $\varepsilon = 2\omega$.

Exercise 32

The figure below shows the vector field for equations (30), which arise from the damped pendulum equation for $0 < \varepsilon < 2\omega$. Describe the behaviour of the pendulum as it follows the path $ABCD$.



Phase diagram for a damped pendulum; the phase paths spiral towards the equilibrium point $(\theta, \dot{\theta}) = (0, 0)$, where the bob hangs downwards

An overview of non-linear differential equations

Non-linear differential equations can describe a vast range of phenomena, and you will find a bewildering number of techniques for treating them discussed in textbooks. But there are three basic approaches that a mathematically trained scientist tries when confronted with non-linear differential equations. These are:

- Consider the equilibrium points of the equations and their stability.
- Try to use geometrical insights, perhaps by making a sketch of the phase paths.
- Use a computer to calculate solutions using numerical methods.

The first two of these approaches have been discussed at length in this unit. Let us make a few comments about how computers are used to solve non-linear equations. The usual tactic is to reduce the problem to a system of first-order differential equations, using the approach discussed in this section. In Unit 2 we explained how first-order differential equations may be solved by using Euler's method, programming a computer to do the repetitive tasks. Euler's method (and the more sophisticated variants that are used in practice) is easily extended to deal with systems of coupled first-order equations.

Of course, there are many approaches tailored to work with particular types of differential equation. But the three approaches mentioned above are the powerful general-purpose tools.

Postscript: what is chaos?

In the Introduction we mentioned that systems described by non-linear differential equations can show a property called 'chaos', which is present in the trajectory shown in Figure 1. We are now able to give a clearer description of what this term means.

In Sections 2 and 3 we discussed the stability of equilibrium points. This idea can be extended to consider the stability of trajectories. Consider a trajectory of a system of equations, represented by a vector $\mathbf{r}(t) = (x(t), y(t), z(t))$. Compare this with a nearby trajectory $\mathbf{r}(t) + \delta\mathbf{r}(t)$, where $\delta\mathbf{r}$ is very small when $t = 0$. We can ask how the separation $s(t) = \delta|\mathbf{r}(t)|$ of two nearby trajectories grows as a function of time t.

It turns out that there are systems where the separation $s(t)$ of trajectories grows like an exponential function of time: $s(t) \simeq S\exp(\lambda t)$, for some positive constants S and λ. A system is said to be *chaotic* if it has this property of '*exponential instability*'.

Chaos is commonly found in systems that have more than two variables, such as the very complicated equations that determine the weather. In 1961 one of the pioneers of this field, Edward Lorenz (1917–2008), was using a computer to model a weather prediction. When, as a shortcut, he entered some data as 0.506 instead of the more precise 0.506 127, he found that the program gave an entirely different prediction. A tiny shift of the starting point of a trajectory had grown into a large separation. He made a very elegant statement of the significance of this discovery: one of his conference presentations (in 1972) was given the title 'Predictability: does the flap of a butterfly's wings in Brazil set off a tornado in Texas?'.

The early papers on chaos theory from the 1960s do not assume much more mathematical knowledge than you have gained from studying this module. And more generally, if you continue to study quantitative sciences, you will find that the topics treated in this module can take you a long way. We hope that you will find some of the material useful, wherever your curiosity takes you in the future.

Learning outcomes

After studying this unit, you should be able to do the following.

- Use a vector field to describe a pair of first-order non-linear differential equations, and use phase paths to represent the solutions.
- Understand the Lotka–Volterra equations modelling the populations of predators and prey, and interpret their solution using phase paths.
- Interpret points and paths in the phase plane.
- Determine whether an equilibrium point is stable or unstable by sketching paths near it.
- Find the equilibrium points for a system of non-linear differential equations.
- Find linear equations that approximate the behaviour of a system of non-linear differential equations near an equilibrium point.
- Use the eigenvalues and eigenvectors of the Jacobian matrix to classify an equilibrium point for a system of linearised equations. Where possible, use this information to classify equilibrium points of the corresponding non-linear equations.
- Describe, using vector fields and phase paths, the qualitative behaviour of the undamped pendulum and the damped pendulum.
- Check that a given quantity is a constant of motion.

Solutions to exercises

Solution to Exercise 1

The growth rate is proportional to the current population when every individual has an equal opportunity to survive and reproduce, and there are no external factors, such as a shortage of food, that might limit growth. This is generally true when the population is relatively small (although when the population is very small, its growth could be limited by difficulties in finding a mate).

Solution to Exercise 2

Assuming a positive constant proportionate growth rate means that no matter how large the population becomes, the proportionate birth rate exceeds the proportionate death rate by the same amount. The population goes on increasing exponentially. This can never be completely realistic for animals in the wild: for example, at some point the food supply that sustains the population must begin to be exhausted. The difference between the proportionate birth and death rates must then fall.

Solution to Exercise 3

The solution of the differential equation $\dot{y} = -hy$ is $y = y_0 e^{-ht}$, where y_0 represents the initial fox population at $t = 0$. So the number of foxes is declining exponentially (because $h > 0$). This decrease is what we would expect as the foxes have no access to their assumed sole source of food, namely rabbits.

Solution to Exercise 4

The system of differential equations is

$$\dot{x} = x, \quad \dot{y} = -y.$$

The general solution is

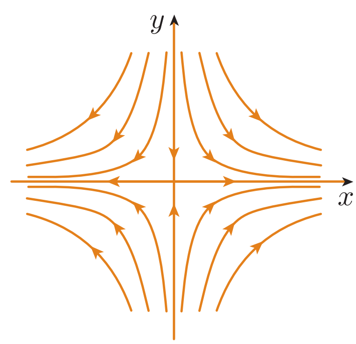
$$x(t) = x_0 e^{t}, \quad y(t) = y_0 e^{-t},$$

where x_0 and y_0 are arbitrary constants. Eliminating t gives $xy = A$, so

$$y = \frac{A}{x},$$

for some constant A. Curves of this form are called *hyperbolas*.

Some paths corresponding to these hyperbolas are shown in the figure in the margin.



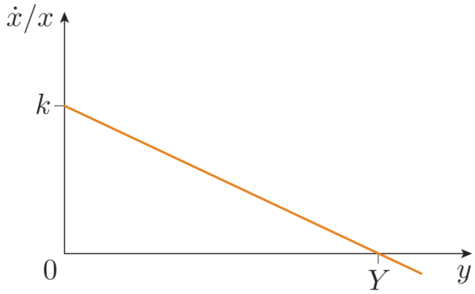
Paths with $x_0 = 0$ and $y_0 \neq 0$ approach the origin as $t \to \infty$, but strictly speaking they never reach it. The only path that includes the origin is the one with $x_0 = y_0 = 0$, which starts at the origin and does not move away from it.

Solution to Exercise 5

Rearranging equation (13), we obtain

$$\frac{\dot{x}}{x} = k - \frac{k}{Y}y,$$

which is the equation of a straight line, as shown below.

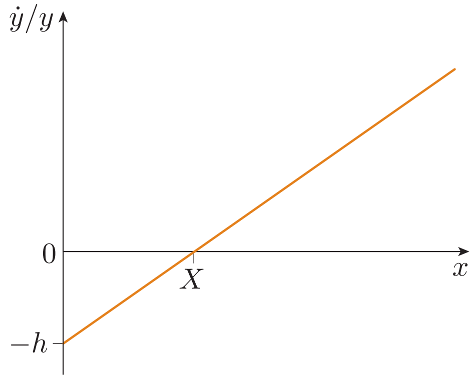


The proportionate growth rate $\dot{x}/x$ of rabbits decreases as the population y of foxes increases, becoming zero when $y = Y$. The population x of rabbits will increase if the population y of foxes is less than Y, but it will decrease if $y > Y$.

Similarly, rearranging equation (14), we obtain

$$\frac{\dot{y}}{y} = -h + \frac{hx}{X}.$$

This is also the equation of a straight line, as shown below.



The proportionate growth rate $\dot{y}/y$ of foxes increases linearly as the population x of rabbits increases. The fox population y will decrease if the population x of rabbits is less than X, but it will increase if $x > X$.

Solution to Exercise 6

(a) $[0 \quad 0]^T$ (Note that $\mathbf{u}(0,0) = \mathbf{0}$.)

(b) $[0 \quad -10]^T$

(c) $[0 \quad -5]^T$

(d) $[50 \quad 0]^T$

(e) $[0 \quad 0]^T$ (Note that $\mathbf{u}(1000,100) = \mathbf{0}$.)

(f) $[0 \quad 5]^T$

(g) $[25 \quad 0]^T$

(h) $[-25 \quad 0]^T$

Solution to Exercise 7

(a) (i) The differential equations under consideration are

$$\dot{x} = 0.05x\left(1 - \frac{y}{100}\right), \quad \dot{y} = -0.1y\left(1 - \frac{x}{1000}\right),$$

with $x \geq 0$ and $y \geq 0$.

In this case, $\dot{x} = 0$ when $x = 0$ or when $y = 100$.

(ii) $\dot{x} > 0$ when $x > 0$ and $0 \leq y < 100$.

(iii) $\dot{x} < 0$ when $x > 0$ and $y > 100$.

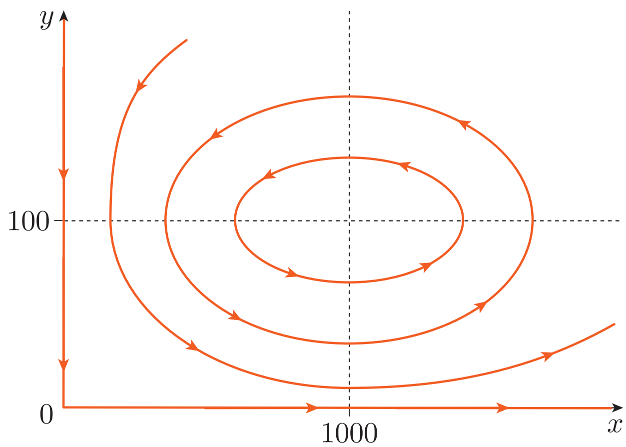
We ignore solutions with $x < 0$ or $y < 0$ because populations cannot be negative.

(b) (i) $\dot{y} = 0$ when $x = 1000$ or when $y = 0$.

(ii) $\dot{y} > 0$ when $x > 1000$ and $y > 0$.

(iii) $\dot{y} < 0$ when $0 \leq x < 1000$ and $y > 0$.

(c) Using the results above together with Figures 9 and 10, typical paths representing solutions are shown below.



This figure shows a path down the positive y-axis, a path to the right along the positive x-axis, and various cycles about the point $(1000, 100)$. The path down the y-axis describes a population of foxes decreasing to zero in the absence of rabbits. The path to the right along the x-axis describes a population of rabbits increasing without limit in the absence of foxes. It will be shown later (in Subsection 3.5) that the cycles are closed (as shown in the diagram), rather than spirals.

Solution to Exercise 8

Using Procedure 1, we have to solve the pair of simultaneous equations

$$0.1x - 0.005xy = 0,$$
$$-0.2y + 0.0004xy = 0.$$

Factorising these equations gives

$$0.1x(1 - 0.05y) = 0,$$
$$-0.2y(1 - 0.002x) = 0.$$

From the first equation, either $x = 0$ or $y = 20$.

If $x = 0$, the second equation gives $y = 0$, hence $(0, 0)$ is an equilibrium point. If $y = 20$, the second equation gives $x = 500$, so $(500, 20)$ is another equilibrium point.

Therefore the only equilibrium points are when there are no animals or when there is a balance between 500 prey and 20 predators. Using the values for this second equilibrium point, the equations can be put in the standard Lotka–Volterra form

$$\dot{x} = 0.1x\left(1 - \frac{y}{20}\right), \quad \dot{y} = -0.2y\left(1 - \frac{x}{500}\right).$$

Solution to Exercise 9

Procedure 1 leads to the pair of simultaneous equations

$$x(20 - y) = 0,$$
$$y(10 - y)(10 - x) = 0.$$

From the first equation, either $x = 0$ or $y = 20$.

If $x = 0$, the second equation gives $y = 0$ or $y = 10$. If $y = 20$, the second equation gives $x = 10$.

Hence the equilibrium points are $(0, 0)$, $(0, 10)$ and $(10, 20)$.

Solution to Exercise 10

(a) Stable. (b) Unstable. (c) Unstable.

Solution to Exercise 11

We evaluate the various partial derivatives given in the solution to Example 3. At the equilibrium point $(0, 0)$, we obtain

$$\frac{\partial u}{\partial x}(0,0) = k, \quad \frac{\partial u}{\partial y}(0,0) = 0,$$
$$\frac{\partial v}{\partial x}(0,0) = 0, \quad \frac{\partial v}{\partial y}(0,0) = -h.$$

Thus the required linear approximation is

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & -h \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix},$$

giving the pair of equations

$$\dot{p} = kp, \quad \dot{q} = -hq.$$

(These are the equations studied in Subsection 1.3.)

Solution to Exercise 12

Here we have

$$u(x,y) = x(20-y),$$
$$v(x,y) = y(10-y)(10-x),$$

giving partial derivatives

$$\frac{\partial u}{\partial x} = 20-y, \quad \frac{\partial u}{\partial y} = -x,$$
$$\frac{\partial v}{\partial x} = -y(10-y), \quad \frac{\partial v}{\partial y} = (10-y)(10-x) - y(10-x) = 2(5-y)(10-x).$$

So the Jacobian matrix of the vector field $\mathbf{u}(x,y)$ is

$$\mathbf{J}(x,y) = \begin{bmatrix} 20-y & -x \\ -y(10-y) & 2(5-y)(10-x) \end{bmatrix}.$$

At the equilibrium point $(10,20)$, we have

$$\mathbf{J}(10,20) = \begin{bmatrix} 0 & -10 \\ 200 & 0 \end{bmatrix},$$

so the linear approximation is

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & -10 \\ 200 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix},$$

giving the pair of equations

$$\dot{p} = -10q, \quad \dot{q} = 200p.$$

Solution to Exercise 13

Solving the equations

$$3x + 2y - 8 = 0,$$
$$x + 4y - 6 = 0,$$

we obtain the equilibrium point $x_{\mathrm{e}} = 2$, $y_{\mathrm{e}} = 1$.

The Jacobian matrix is

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}.$$

In this case, the elements of the Jacobian matrix are all constants, so putting $x = 2 + p$ and $y = 1 + q$, we obtain the matrix equation

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

The corresponding system of equations is

$$\dot{p} = 3p + 2q,$$
$$\dot{q} = p + 4q.$$

Solution to Exercise 14

(a) To find the equilibrium points, we solve the simultaneous equations

$$0.5x - 0.000\,05x^2 = 0,$$
$$-0.1y + 0.0004xy - 0.01y^2 = 0.$$

Factorising these equations gives

$$0.5x(1 - 0.0001x) = 0,$$
$$-0.1y(1 - 0.004x + 0.1y) = 0.$$

The first equation gives

$$x = 0 \quad \text{or} \quad x = 10\,000.$$

If $x = 0$, the second equation is

$$-0.1y(1 + 0.1y) = 0,$$

which gives $y = 0$ or $y = -10$. As $y \geq 0$, only the first solution is possible. This leads to the equilibrium point $(0, 0)$.

If $x = 10\,000$, the second equation is

$$-0.1y(-39 + 0.1y) = 0,$$

which gives $y = 0$ or $y = 390$. So we have found two more equilibrium points, namely $(10\,000, 0)$ and $(10\,000, 390)$.

(b) We have

$$u(x, y) = 0.5x - 0.000\,05x^2,$$
$$v(x, y) = -0.1y + 0.0004xy - 0.01y^2.$$

So the Jacobian matrix is

$$\mathbf{J}(x, y) = \begin{bmatrix} 0.5 - 0.0001x & 0 \\ 0.0004y & -0.1 + 0.0004x - 0.02y \end{bmatrix}.$$

(c) At the equilibrium point $(0, 0)$,

$$\mathbf{J}(0, 0) = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.1 \end{bmatrix},$$

and the linearised approximations to the differential equations in the neighbourhood of this equilibrium point are

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

At the equilibrium point $(10\,000, 0)$,

$$\mathbf{J}(10\,000, 0) = \begin{bmatrix} -0.5 & 0 \\ 0 & 3.9 \end{bmatrix},$$

and the linearised approximations to the differential equations near this equilibrium point are

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & 3.9 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

Finally, at the equilibrium point $(10\,000, 390)$,

$$\mathbf{J}(10\,000, 390) = \begin{bmatrix} -0.5 & 0 \\ 0.156 & -3.9 \end{bmatrix},$$

and the linearised approximations to the differential equations near the equilibrium point are

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0.156 & -3.9 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

Solution to Exercise 15

(a) The characteristic equation of the Jacobian matrix is

$$(3 - \lambda)(1 - \lambda) = 0,$$

so the eigenvalues are $\lambda = 3$ and $\lambda = 1$.

(b) As the eigenvalues are positive and distinct, the equilibrium point is a *source*, and is unstable.

Solution to Exercise 16

(a) The characteristic equation of the Jacobian matrix is

$$-\lambda(-3 - \lambda) + 2 = 0,$$

i.e. $\lambda^2 + 3\lambda + 2 = 0$, which factorises to give

$$(\lambda + 1)(\lambda + 2) = 0,$$

so the eigenvalues are $\lambda = -1$ and $\lambda = -2$.

(b) As the eigenvalues are negative and distinct, the equilibrium point is a *sink* (which is stable).

Solution to Exercise 17

(a) The differential equations are

$$\dot{p} = 2p,$$
$$\dot{q} = 2q,$$

which have general solution

$$p(t) = Ce^{2t}, \quad q(t) = De^{2t},$$

where C and D are arbitrary constants.

(b) Eliminating t from the general solution, the equations of the paths are

$$q = \frac{D}{C}p = Kp,$$

where $K = D/C$ is also an arbitrary constant. So the paths are all straight lines passing through the origin.

The above analysis has neglected the possibility $C = 0$. In this case the path is $p = 0$, which is also a straight line passing through the origin, namely the q-axis.

(c) The magnitudes of both $p(t)$ and $q(t)$ are increasing functions of time, so the point $(p(t), q(t))$ moves away from the origin as t increases. So the equilibrium point is *unstable*.

Solution to Exercise 18

(a) The characteristic equation of the matrix is

$$(1-\lambda)(-2-\lambda)-4=0,$$

i.e. $\lambda^2+\lambda-6=0$, which factorises to give

$$(\lambda-2)(\lambda+3)=0,$$

so the eigenvalues are $\lambda=2$ and $\lambda=-3$.

The eigenvector $[a \quad b]^T$ corresponding to $\lambda=2$ satisfies the equation

$$\begin{bmatrix}-1 & 2\\ 2 & -4\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix},$$

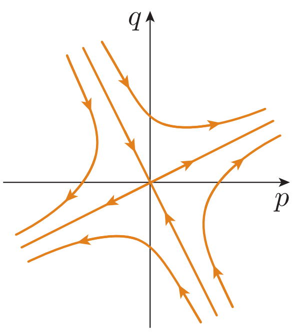
so $-a+2b=0$, and an eigenvector corresponding to the positive eigenvalue $\lambda=2$ is $[2 \quad 1]^T$. Other eigenvectors corresponding to $\lambda=2$ are multiples of this; all these eigenvectors are along the line $q=\frac{1}{2}p$.

The eigenvector $[a \quad b]^T$ corresponding to $\lambda=-3$ satisfies the equation

$$\begin{bmatrix}4 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix},$$

so $4a+2b=0$, and an eigenvector corresponding to the negative eigenvalue $\lambda=-3$ is $[1 \quad -2]^T$; all these eigenvectors are along the line $q=-2p$.

(b) The matrix has one positive and one negative eigenvalue, so the (unstable) equilibrium point is a *saddle*.



(c) There are two straight-line paths, namely $q=\frac{1}{2}p$ and $q=-2p$, corresponding to the two eigenvectors. On the line $q=\frac{1}{2}p$, the point $(p(t),q(t))$ moves away from the origin as t increases, because the corresponding eigenvalue is *positive*. On the line $q=-2p$, the point approaches the origin as t increases, because the corresponding eigenvalue is *negative*. This information, together with the knowledge that the equilibrium point is a saddle, allows us to sketch the phase diagram in the margin.

Solution to Exercise 19

(a) The characteristic equation of the Jacobian matrix is

$$(2-\lambda)(-2-\lambda)+5=0,$$

i.e. $\lambda^2+1=0$, so the eigenvalues are $\lambda=i$ and $\lambda=-i$.

(b) As both of the eigenvalues are imaginary, the equilibrium point is a *centre*, which is stable.

Solution to Exercise 20

(a) The characteristic equation of the Jacobian matrix is

$$(1-\lambda)^2+1=0,$$

i.e. $\lambda^2-2\lambda+2=0$, which has complex roots $\lambda=1+i$ and $\lambda=1-i$.

(b) As the eigenvalues are complex with positive real part, the equilibrium point is a *spiral source* (and is unstable).

Solution to Exercise 21

(a) The point could be a source, a star source or a spiral source (all of which are unstable).

(b) The point could be a sink, a star sink or a spiral sink (all of which are stable)

Solution to Exercise 22

The Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} 0 & -kX/Y \\ hY/X & 0 \end{bmatrix},$$

which has the characteristic equation

$$\lambda^2 + hk = 0.$$

The eigenvalues are $\lambda = \pm i\sqrt{hk}$, which are purely imaginary, so the equilibrium point is a *centre*, which is stable.

Solution to Exercise 23

The eigenvalues of the Jacobian matrix are $\lambda = k$ and $\lambda = -h$, which are real and have opposite signs because $k > 0$ and $h > 0$. So the equilibrium point is a *saddle*. (In fact, in this case we have to restrict p and q to non-negative values, but this does not affect our conclusion.)

Solution to Exercise 24

The characteristic equation is $(2-\lambda)(2-\lambda) + 9 = 0$, which gives

$$\lambda^2 - 4\lambda + 13 = 0.$$

The eigenvalues are

$$\lambda = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i.$$

The eigenvector $[a \quad b]^T$ corresponding to the eigenvalue $2+3i$ satisfies

$$\begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so $-3ia - 3b = 0$. Hence an eigenvector corresponding to $2+3i$ is $[1 \quad -i]^T$. (In a similar way, an eigenvector corresponding to $2-3i$ is $[1 \quad i]^T$.)

To find the general solution of the system of differential equations, we need to find the real and imaginary parts of $[1 \quad -i]^T e^{(2+3i)t}$. We get

$$\begin{aligned} e^{2t}e^{3it}\begin{bmatrix} 1 \\ -i \end{bmatrix} &= e^{2t}(\cos 3t + i\sin 3t)\begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &= e^{2t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + ie^{2t}\begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix}. \end{aligned}$$

The general solution is

$$\begin{bmatrix} p \\ q \end{bmatrix} = Ce^{2t} \begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + De^{2t} \begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix}.$$

As the eigenvalues are complex with a positive real component, the equilibrium point $p = 0$, $q = 0$ is a *spiral source*, which is unstable (see Figure 19).

As the equilibrium point of the linear approximation is not a centre, the corresponding equilibrium point of the non-linear system is also a spiral source.

Solution to Exercise 25

(a) The equilibrium points are given by

$$(1 + x - 2y)x = 0,$$
$$(x - 1)y = 0.$$

The second equation gives

$$x = 1 \quad \text{or} \quad y = 0.$$

When $x = 1$, substituting into the first equation gives

$$2 - 2y = 0,$$

which leads to $y = 1$. So $(1, 1)$ is an equilibrium point.

When $y = 0$, substituting into the first equation gives

$$(1 + x)x = 0,$$

hence $x = 0$ or $x = -1$. So we have found two further equilibrium points, namely $(0, 0)$ and $(-1, 0)$.

(b) With the usual notation,

$$u(x, y) = (1 + x - 2y)x = x + x^2 - 2xy,$$
$$v(x, y) = (x - 1)y = xy - y.$$

So the Jacobian matrix is

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 + 2x - 2y & -2x \\ y & x - 1 \end{bmatrix}.$$

(c) At the point $(0, 0)$, the Jacobian matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

so the linearised system is

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

The eigenvalues of the Jacobian matrix are $\lambda = 1$ and $\lambda = -1$. As one of the eigenvalues is positive and the other is negative, the equilibrium point of the linearised system is a *saddle*.

At the point $(-1, 0)$, the Jacobian matrix is

$$\begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix},$$

so the linearised system is

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix is $(-1-\lambda)(-2-\lambda) = 0$, so the eigenvalues are $\lambda = -1$ and $\lambda = -2$. As these eigenvalues are negative and distinct, the equilibrium point of the linearised system is a *sink*.

At the point $(1, 1)$, the Jacobian matrix is

$$\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix},$$

so the linearised system is

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix is

$$(1-\lambda)(-\lambda) + 2 = \lambda^2 - \lambda + 2 = 0.$$

The roots of this quadratic equation are

$$\lambda = \tfrac{1}{2}(1 \pm i\sqrt{7}),$$

so the eigenvalues are complex with a positive real part.

Hence the equilibrium point of the linearised system is a *spiral source*.

(d) As none of the equilibrium points of the linearised systems found in part (c) are centres, the behaviour of the original non-linear system near the equilibrium points is the same as that of the linear approximations. In other words,

$(0, 0)$	is a *saddle*, which is unstable,
$(-1, 0)$	is a *sink*, which is stable,
$(1, 1)$	is a *spiral source*, which is unstable.

Solution to Exercise 26

If we replace θ by x and let $y = \dot{x} = \dot{\theta}$, then we have

$$\dot{y} = \ddot{x} = \ddot{\theta} = -\omega^2 \sin\theta - \varepsilon\dot{\theta} = -\omega^2 \sin x - \varepsilon y.$$

So the system of first-order differential equations that is equivalent to equation (27) is

$$\dot{x} = y,$$
$$\dot{y} = -\omega^2 \sin x - \varepsilon y.$$

The associated vector field is

$$\mathbf{u}(x, y) = \begin{bmatrix} y \\ -\omega^2 \sin x - \varepsilon y \end{bmatrix}.$$

Solution to Exercise 27

(a) To find the equilibrium points, we use Procedure 1 and put $\mathbf{u}(x, y) = \mathbf{0}$. This gives

$$y = 0,$$
$$-\omega^2 \sin x = 0.$$

So there are two equilibrium points in the range $-\pi < x \leq \pi$, namely $(0, 0)$ and $(\pi, 0)$.

(b) The equilibrium point $(0, 0)$ corresponds to $x = 0$, $\dot{x} = 0$. Physically, this corresponds to a stationary pendulum hanging vertically downwards. From experience, we know that a small disturbance from this equilibrium point will result in small oscillations about the downwards vertical. So we would expect this equilibrium point to be *stable*.

The equilibrium point $(\pi, 0)$ corresponds to $x = \pi$, $\dot{x} = 0$, which is a pendulum pointing vertically upwards at rest. (Not easy to achieve in practice!) A small disturbance from this equilibrium point will result in the pendulum moving away from the upwards vertical and speeding up until it is vertically downwards. It will then move through its lowest position and continue to move in the same direction, slowing down and heading towards the highest point. So we would expect this equilibrium point to be *unstable*.

Solution to Exercise 28

(a) As in the solution to Exercise 27(a), to find the equilibrium points we need to find the solutions of

$$y = 0,$$
$$-\omega^2 \sin x - \varepsilon y = 0.$$

Substituting $y = 0$ from the first equation into the second equation leads to $x = 0$ or $x = \pi$. So there are two equilibrium points in the range $-\pi < x \leq \pi$, namely $(0, 0)$ and $(\pi, 0)$.

(b) Using reasoning similar to that used in the solution to Exercise 27(b), the equilibrium point $(0, 0)$ corresponds to a pendulum hanging vertically downwards at rest. We expect this equilibrium point to be stable. The equilibrium point $(\pi, 0)$ corresponds to a stationary pendulum pointing vertically upwards. As in Exercise 27, we expect this equilibrium point to be unstable.

Solution to Exercise 29

(a) Using the usual notation,

$$u(x, y) = y,$$
$$v(x, y) = -\omega^2 \sin x.$$

So the Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos x & 0 \end{bmatrix}.$$

(b) Using Procedure 2, in the neighbourhood of the equilibrium point $(0, 0)$, the linear system of differential equations that approximates the non-linear system is

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix is

$$\lambda^2 + \omega^2 = 0,$$

so the eigenvalues are $\lambda = \pm i\omega$.

Hence the equilibrium point $(0, 0)$ is a *stable centre* of the linearised system.

The linearised system of differential equations in the neighbourhood of the equilibrium point $(\pi, 0)$ is

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

The eigenvalues of the Jacobian matrix are $\lambda = \pm\omega$.

The eigenvalues have opposite signs, so the equilibrium point $(\pi, 0)$ is a *saddle* of the linearised system.

Solution to Exercise 30

The equations of motion are $\dot{x} = y$, $\dot{y} = -\omega^2 \sin x$, and we have

$$\frac{\partial E}{\partial x} = \omega^2 \sin x, \quad \frac{\partial E}{\partial y} = y,$$

so equation (23) gives

$$\begin{aligned} \frac{dE}{dt} = \frac{\partial E}{\partial x}\,\dot{x} + \frac{\partial E}{\partial y}\,\dot{y} &= \omega^2 \sin x \dot{x} + y\dot{y} \\ &= \omega^2 (\sin x) y - y\omega^2 \sin x = 0. \end{aligned}$$

So $E(t) = E(x(t), y(t))$ is indeed a constant of motion.

Solution to Exercise 31

(a) Using the usual notation,

$$\begin{aligned} u(x, y) &= y, \\ v(x, y) &= -\omega^2 \sin x - \varepsilon y. \end{aligned}$$

So the Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos x & -\varepsilon \end{bmatrix}.$$

(b) The linear system of differential equations that approximates the system near the equilibrium point $(0, 0)$ is

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\varepsilon \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix is

$$-\lambda(-\varepsilon - \lambda) + \omega^2 = \lambda^2 + \varepsilon\lambda + \omega^2 = 0.$$

For $0 < \varepsilon < 2\omega$, the eigenvalues are

$$\lambda = \tfrac{1}{2}\left(-\varepsilon \pm i\sqrt{4\omega^2 - \varepsilon^2}\right),$$

so the equilibrium point is a *spiral sink* (which is stable).

For $\varepsilon > 2\omega$, the eigenvalues are

$$\lambda = \tfrac{1}{2}\left(-\varepsilon \pm \sqrt{\varepsilon^2 - 4\omega^2}\right).$$

Both the eigenvalues are negative, so $(0, 0)$ is a *sink* (which is stable).

The linearised system of differential equations in the neighbourhood of the equilibrium point $(\pi, 0)$ is

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega^2 & -\varepsilon \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix is

$$-\lambda(-\varepsilon - \lambda) - \omega^2 = \lambda^2 + \varepsilon\lambda - \omega^2 = 0,$$

so the eigenvalues are

$$\lambda = \tfrac{1}{2}\left(-\varepsilon \pm \sqrt{\varepsilon^2 + 4\omega^2}\right).$$

One of these eigenvalues is positive, whereas the other is negative, so the equilibrium point $(\pi, 0)$ is a *saddle* (which is unstable).

Solution to Exercise 32

At A, $y = \dot{\theta} > 0$ so the pendulum is moving in an anticlockwise direction and it is approaching its highest point ($x = \pi$). It slows down as it passes through this point, and continues to slow down until a little after the highest point at B. It then continues to move in an anticlockwise direction, and speeds up until it reaches C. It moves through its lowest position ($x = 2\pi$), still moving anticlockwise, and heads towards its highest point again – but does not reach it. At D it stops, then falls back and oscillates about its lowest point with ever-decreasing amplitude.

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